

DESIGN COMPARISONS AND MODELING ASPECTS
FOR UNBALANCED-RANDOM MODELS

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To God for grace and strength

my people

and

my wife

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The quality of estimation of variance components depends not only on the choice of the method of analysis but also on the design used to generate the response data. Methods for comparing designs for a linear model have focused on criteria based on single-sided functions. However, these functions are, in general, difficult to use due to their complex forms in addition to their dependency on the model's unknown variance components. By using quantiles of an estimator of a function of variance components, we can assess the overall quality of estimation. We can also compare designs for estimating variance components. In this dissertation, we present a graphical approach which provides a comprehensive picture of the quality of estimation obtained with a given design. This approach can also be used to compare several candidate designs. Designs for one-way, two-way cross classification without interaction, and three-fold nested random models are considered. Two methods of estimation of variance components, namely analysis of variance (ANOVA) and maximum likelihood (ML) methods, are investigated.

Adhering balance in a data set is a desirable feature since estimators and tests concerning the variance components of a given model have optimal properties, for example under the normality assumption. Complete data balance, however, is not always attainable. Therefore, the effect of imbalance of a design on a particular quantity of interest is always a concern to an experimenter. In this dissertation, we study the overall effect of data imbalance, as well as the effect of variance components, using a method for generating designs having specified degrees of imbalance. The probability of getting a unique ANOVA estimate of a variance component, the power of an F -test concerning a variance component, with or without homogeneity assumption of error variances, and the coverage probability of a confidence interval on a variance component are considered as quantities of interest. Generalized linear models techniques are used to model a quantity of interest in terms of factors considered to have imbalance on such a quantity. This modeling scheme allows us to assess a better insight into the effect of imbalance on the quantity of interest under consideration.

CHAPTER 1 INTRODUCTION

In an experimental situation, the levels of a factor may not be subject to the control of the experimenter, or they may be selected at random. The effect of such a factor is considered random, and its variance is termed variance component. Furthermore, data sets obtained in an experiment may be unbalanced. By this it is meant that the numbers of observations at the various sublevels of the data are not all equal. The pattern that these numbers take in a given experimental situation is referred to as a design. In particular, if these numbers are equal then the data set is called balanced, and the associated design is said to be completely balanced. The design used in connection with a given model plays an important role in assessing the significance of the various effects in the model. This is particularly true in the analysis of the model's variance components.

Several designs are often possible for a given set of experimentation. Determination of optimal designs and estimation of parameters of a fitted model are therefore needed. The choice of a design and the corresponding method of analysis usually depend on which functions of variance components are estimated. While much has been done in the area of estimation of variance components, design aspects associated with the analysis of variance components are less developed. This discrepancy arises in part from the lack of a general methodology to compare designs. In several studies, designs were compared using various criteria. Most of these criteria, however, employed a single-valued function such as the trace or determinant of the variance-covariance matrix of the vector of variance component estimators.

Achieving balance in a data set is a desirable feature due to the optimal properties associated with estimators and tests concerning the variance components of a given model under the normality assumption. Complete data balance, however, is not always attainable. Therefore, there is a need to have at our disposal adequate procedures for the analysis of unbalanced data. The analysis of unbalanced models is quite complicated. Unlike balanced models, there is no unique analysis of variance (ANOVA) since the partitioning of the total sum of squares can be done in a variety of ways. Moreover, the distributional properties of its components and independence of the sums of squares are no longer valid. As a result, optimal properties of the ANOVA estimators, namely minimum variance quadratic unbiasedness and minimum variance unbiasedness under normality are lost (see Searle, Casella, and McCulloch, 1992, pp. 225-232). Therefore, whenever we have an unbalanced data set, questions arise as to how unbalanced the data set is, what effects does imbalance (or imbalance) have on data analysis. To answer the first question, we need a certain measure for determining the degree of departure from balance. With this measure, we can quantify the degree of imbalance rather than using subjective descriptions such as "extremely unbalanced," "moderately unbalanced," or "nearly balanced." Kiers (1987, p. 385) stated that such a measure can serve "as an indicator of the suitability of approximate methods that are adopted from balanced-data-based procedures and used to analyze an unbalanced model."

Unbalanced designs present difficulties in comparative studies of methods for estimating variance components. Even the simple idea of comparing sampling variances is unworkable because the variances themselves are not only functions of the variance components but also functions of the design. D (Searle, Casella, and McCulloch, 1992, p. 232). In this context, the non-difficulty in designing studies here "adequately cover all possible designs" so as to provide a basis for drawing general conclusions. One may observe a collection of patterns of cell frequencies "which initially seem to suggest

from slightly to badly unbalanced" (Gendow and Steiner, 1979, p. 175). However, as will be seen later, having a method for measuring the degree of imbalance, coupled with the ability to generate designs with a specified degree of imbalance, will enable us to adequately study the effect of imbalance on particular quantities of interest. For example, the effect on the efficiency of parameter estimation and on tests of equal interest are necessarily ignored. In addition, we can also perform better comparisons among methods of estimation of variance components and among designs (see Khuri, Mathew and Sacks, 1996, Ch. 1, pp. 55-66).

In this dissertation, we have concentrated on comparing designs for estimating variance components using a graphical approach: the so-called quantum dispersion graphs (QDGs) which was first introduced in Khuri (1987). We have also investigated the effect of imbalance on various quantities of interest which are modeled using generalized linear models techniques. For this, the method of generating designs with a specific degree of imbalance is introduced in Khuri (1991) as well.

This dissertation is divided into five chapters. Chapter 1 gives a brief introduction to the area of design components and the study of the effect of imbalance. In Chapter 2, we review the previous statistical literature. A related survey is found in Khuri and Sacks (1995, pp. 366-394). Chapter 3 is devoted to comparing designs for the one-way random model, which is in Section 3.1 for the two-way cross class fixation model, which is in Section 3.2, and for the three-fold nested random model, which is in Section 3.4, using the QDGs and the so-called empirical quantum dispersion graphs (EQDGs). In each case, five estimation methods, namely, ANOVA and RE, are considered to compare these estimation methods for a given design. In Chapter 4, an investigation is made concerning the effect of imbalance on various quantities of interest, including the probability of a negative ANOVA estimate of a variance component in Section 4.1; the power of an F test for a variance component

under homogeneous error variances in Section 4.2, the power of an F test for a variance component under heterogeneous error variances in Section 4.3, and the coverage probability of a confidence interval for a variance component in Section 4.4. Finally Chapter 5 presents a discussion of future research areas. The program for generating designs with a specified degree of imbalance, which is developed for this research, is provided in the Appendix.

CHAPTER 3 LITERATURE SURVEY

3.1. Design Considerations

A study study in design for the one-way random model was carried out by Crump (1964), whose results were later published by Anderson and Crump (1967). Anderson and Crump derived an optimal allocation procedure to estimate σ_e^2 (the among-class variance component) or the ratio $\eta = \frac{\sigma_e^2}{\sigma_b^2}$ (σ_b^2 is the within-class variance component). These procedures minimize the variances of the ANOVA estimates, s_e^2 and $\hat{\eta}$, for a fixed number of groups (k) and a fixed total sample size (N). They also presented procedures for determining the optimal choice of k to estimate σ_e^2 and η . They defined the efficiency of estimating σ_e^2 or η as the ratio of the variance of the estimate with an optimal k ($\eta =$ an optimal design) to that of the estimate at a given k ($\eta =$ a given design). They showed that a variable choice of the sampling plan (in terms of k) can provide large gains in the efficiencies of the estimates, s_e^2 and $\hat{\eta}$.

Thompson and Anderson (1974) compared several balanced designs, for the one-way random model, using ANOVA estimates, truncated ANOVA estimates, maximum likelihood (ML) estimates, and modified maximum likelihood (MML) estimates of the variance components. The last estimates were considered by Klotz, Milton, and Zacks (1966). Thompson and Anderson also considered an unbalanced design which allocates one observation to each of k_1 classes and two observations to each of $k_2 (= k - k_1)$ classes. Such a design was referred to as a (k_1, k_2) -design. This design was found to be optimal for estimating σ_e^2 when $\eta > 1$ by Anderson and

Group (1967). Thompson and Anderson found that, regardless of how large η was, ML provided an estimator of σ_j^2 with a smaller asymptotic variance than the procedures of choosing iteratively weighted ANOVA and iteratively unweighted ANOVA estimators. However, the difference between the unweighted ANOVA and the ML results was small when η was large.

Monte-Carlo-based comparisons of estimators of variance components with non balanced designs were made by Swales and Monden (1984), and Chakraborty (1987). Both papers considered one-way random model with an arbitrarily chosen unbalanced design. Chakraborty showed that Bayesian estimators were viable competitors to many classical alternative estimators.

Comparisons of designs for the two-way cross classification random model are considered in Gayler (1983). Substantial modifications in this area, however, were made by H. L. Anderson and his colleagues. Hsu and Anderson (1978) compared several particular designs using ML in a model without interaction. The asymptotic behavior of ML estimators of variance components was examined through its large sample variance-covariance matrix. They concluded that prior information was necessary for selecting a good design for the estimation of variance components. Using the trace of an asymptotically variance-covariance matrix as a criterion, they also found that the balanced design was best only when σ_c^2 is the dominant variance component, that is, when σ_c^2 is larger than both σ_r^2 and σ_j^2 , the variance components for rows and columns, respectively. Conclusions favoring certain unbalanced designs over balanced designs can also be found in Monden (1987). Monden considered the two-way crossed classification random model with interaction.

Hsu, Anderson, and Chakraborty (1982) extended the result of Hsu and Anderson (1978) to include additional comparisons of designs for the two-way cross classification random model with interaction using asymptotic maximum likelihood

results. Similar findings and results, as those in Meier and Anderson [1973], were reported.

Levene and Nelson [1955] employed the balanced three-fold nested random model. They conducted empirical studies of the sampling distributions of ANOVA estimates using eight different combinations of variance components sets. Histograms of σ^2_{α} under the normally rectangular were provided. They also showed empirical and analytical results concerning percentages of negative estimates of variance components.

Levene, Nelson, Johnson, and Klotz [1955] considered the unbalanced three-fold nested random model. They employed the "staggered" and "overlaid" four stage nested designs and presented empirical results concerning the distributions of the variance components for both designs with eight sets of variance components. The normal, rectangular, and exponential underlying distributions were considered. No particular difference between staggered and overlaid designs was detected on the basis of the empirical shape of the distributions of the ANOVA estimates of σ^2_{α} . With the normal and rectangular underlying distributions, variance components estimates from the first three stages behaved quite similarly to their expected distributions. However, the exponential distributions produced negative variance estimates because of its long tail. Since Raudenbush (1952, 1953) introduced the staggered design, it has been received widespread attention by several researchers. (See Smith and Brerly, 1944; Nelson, 1952, 1955a, 1955b; Klotz and Volk, 1955; Klotz, Volk, and Meier, 1957)

Caldwell and Gayler [1955] considered the two-fold nested random model with variance components σ^2_{α} , $\sigma^2_{\beta|\alpha}$, and σ^2_{ϵ} . They studied the effect of designs on the percent of the ANOVA estimates of these variance components using computer simulation. A total of 14 unbalanced designs were chosen, and for each design, 40 different variance components configurations of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and 10 different matrix ones were considered. Three criteria were proposed to compare designs and to determine

optimal designs over the variance component configurations and sample sizes. These criteria were based on the trace, the determinant, and the weighted trace of the variance-covariance matrix of the vector of variance-component estimators. The trace criterion was recommended since it was sensitive to changes in the sample size and the variance-component configuration, as well as being the easiest criterion to compute. When both $\eta_1 = \frac{C_1^2}{C_2^2}$ and $\eta_2 = \frac{C_1}{C_2}$ were small, a balanced design was found to be optimal in the sense of providing the smallest value of the trace. When η_1 was large, designs with the highest second-stage degree of freedom were favorable, while η_2 was large, designs with high first-stage sampling were selected.

Cunningham and Dayter (1974) considered variance component testing for an unbalanced two-fold nested random model. An exact test for the hypothesis of $H_0: \sigma_{\beta\gamma}^2 = 0$ can be formed regardless of unbalancedness of the design, but not for the hypothesis of $H_0: \sigma_{\gamma}^2 = 0$. It is because, with an unbalanced design, the properties of independence and chi-squaredness of mean squares no longer hold under the usual normality assumptions. Ignoring this failure of properties, they employed Satterthwaite's approximation to test $H_0: \sigma_{\gamma}^2 = 0$. This test is normally used when an exact test is unavailable in balanced ANOVA tables. They considered "test size distortions" as the discrepancy of the estimated size of this approximate test from the assumed test size α . Dependence and non-chi-squaredness of a mean square are considered as confounding factors for the test size distortions. They noticed that, in the case of $n_{1j} = n_{2j}$, $j \neq j-1 = 1, 2, \dots, J$ (Bland (1977) used the terminology "partially balanced mixed design" for this case), independence of the mean squares is obtained, but not chi-squaredness. In this situation, the test size distortions increased as η_1 increased. On the other hand, when designs have chi-squared, but dependent mean squares, the test size distortions decreased as η_1 increased with relatively the mean

magnitude. When both properties of independence and the homoskedastic were violated, the test size disturbance was apparently small, a behavior which Casagrande and Glynn [1974, p. 709] attributed to a so-called "omnibus-balancing effect."

Tan and Cheng [1984] considered the problem of testing a variance component in an unbalanced two-field nested random model. They employed two types of unbalanced designs in order to measure and compare their test statistics for $H_0: \sigma^2 = 0$ with respect to power of the test. They obtained approximate power functions using Lagrange polynomial equations.

Mukerjee and Rydz [1988] considered multifactor random effects model to estimate variance components by the method of unweighted square of mean. They showed that, when the numbers of levels of the factors are considered fixed, a balanced design, if it exists, is optimal in the sense of maximizing variances of the variance components.

Khuri [1987] obtained the exact distribution of an ANOVA estimator of a variance component by deriving an quantiles in order to assess the quality of ANOVA estimators. He demonstrated this using the balanced two-way cross classification random model with interaction. Since the expected mean squares in the corresponding ANOVA table are functions of the variance components, the exact distribution of a scaled value of an ANOVA estimator, namely $W = \frac{Q}{\sigma^2}$, can be expressed as a linear combination of independent chi-squared variates, and as a function of the following series of variance components: $\theta_1 = \frac{\sigma^2}{\sigma^2 + \sigma_{\alpha}^2}$, $\theta_2 = \frac{\sigma^2}{\sigma^2 + \sigma_{\beta}^2}$. Using the algorithm given by Durbin [1983], the quantiles of W , namely $q(p)$ such that $P(W \leq q(p)) = p$, are calculated for each $p \in (0, 1)$. The distribution of W can be depicted using a quantile plot of W for the given design and the actual values of θ_1 and θ_2 . Quantile plots provide information concerning the efficiency of $\hat{\sigma}^2$ by measuring the spread in the values of W , the extent to which an ANOVA estimator can be negative, and the stability in the values of W . Khuri also proposed the so-called quantile dispersion graphs (QDGs)

not only as a means for assessing the overall quality of ANOVA estimation for a particular design, but also as an effective tool to compare the capability of competing designs for recovering variance components. Maximum and minimum quantiles of $\hat{V}$ were obtained over a set of several values of θ and δ_0^2 selected inside a region in the parameter space of θ . The Q/Qn were constructed by plotting these maximum and minimum quantile values against ρ . Features of this methodology, however, is needed to effect estimation of a variance component, e.g. REML and REML estimation.

2.2. Unbalanced Data Imbalance

Law (1978) implicitly mentioned that the quantity $\sum_{i=1}^I(n_i - \bar{n})^2$ could measure deviation from complete balance for the one-way model since it would be small for nearly equal n_i 's. Here, n_i denotes the number of observations from the i^{th} treatment, and $\bar{n} = \frac{1}{I} \sum_{i=1}^I n_i$. Here (1978), however, was the first to propose measuring imbalance. Shortly extending Law's idea, in his study of the sensitivity of maximum-likelihood quadratic unbiased estimation with respect to the α priori weights for σ_1^2 and σ_2^2 , the model's variance components,

Abram and Placet (1981) made the first formal attempt to quantify the degree of imbalance, developing two measures of imbalance for the one-way model (k is the number of groups):

$$r(D) = \frac{D^2}{\sum_{i=1}^I n_i \sum_{i=1}^I \frac{1}{n_i}} \quad \text{and} \quad v(D) = \frac{(\sum_{i=1}^I \frac{1}{n_i})^2}{k \sum_{i=1}^I \frac{1}{n_i}},$$

where D denotes a design, that is, a set of n_i values for the one-way model. The first measure is such that $0 < r(D) \leq 1$, whereas for the second measure, $\frac{1}{k} < v(D) \leq 1$. Large values of these measures indicate "near balance", while values close to the respective lower bounds indicate severe imbalance. Abram and Placet related these

measures to assess the efficiency of an unbalanced design as compared to a balanced design with the same number of observations.

Klein (1987) proposed a measure of data imbalance as a function of Pearson's chi-squared statistic, X^2 , for measuring departure of the observed frequencies of the data from "complete balance". In addition to complete balance, where frequencies are equal within all the subgroups, he also considered certain types of "imbalance", such as "proportional subclass frequencies" in cross-classification models, "partial balance" and "last-stage uniformity" in nested models. His procedure measures the degree of imbalance in the following way: based on an appropriate loglinear model for the subclass frequencies, ML estimates of the expected frequencies are calculated for each type of "balance". Using these estimates, Pearson's approximate chi-squared statistic, which is used in the goodness-of-fit test for the loglinear model, is obtained. A simple function of that statistic leads to the measure of data imbalance, which quantifies departure from the type of "balance" considered. For the one-way model, the measure, $B(D)$, proposed by Klein coincides with Adelman and Fienberg (1982) measure, $e(D)$. The generalization of the procedure proposed by Klein for measuring data imbalance in higher order models is straightforward.

Lera Marques (1994) extended the measures of imbalance given by Adelman and Fienberg (1982) to higher order models. She used an arithmetic average – without providing any rationale for this extension – of Klein and Fienberg's measure obtained from each stage (for nested models), or from each row and column (for cross-classification models). She illustrated her measures using four models, namely, two-fold, three-fold nested models, two-way and three-way cross classification models. Since an unweighted average of the measures of imbalance does not fully convey the information of data structure, Lera's procedure should be modified by using weighted averages.

3.1 Generation of Designs and an Investigation of the Effect of Data Imbalance

To study the effect of imbalance on data analysis, several quantities of interest have been considered. The following is a selected list of these quantities: the coverage probability of a confidence interval for a variance component [Thomas and Fienberg, 1975], the ratio of mean squared error (MSE) of minimum norm quadratic (MNQ) estimators for completely balanced and unbalanced designs [Adams and Searles, 1961] (also, sampling variance and MSE of ANOVA and ML estimators of variance components [Cox, Greenman and Fienberg, 1985], large sample relative efficiencies in the ratio of variance of ANOVA estimators to those of ML estimators of σ_e^2 and σ_i^2 for the one-way model [Dawson and Kowal, 1985], empirical power of the F -test and the likelihood ratio(LR)-test of σ_e^2 in the one-way random model [Dawson and Kowal, 1985], the relative efficiency in the ratio of the variance of an ANOVA estimator to that of minimum norm quadratic unbiased (MNQU) estimator of σ_e^2 [Adams and Searles, 1961], the scaled variance of an ANOVA estimator of σ_e^2 [Cox, 1950] and the ratios of empirical power of tests for σ_e^2 [El Shattawi and Bely, 1952]. Except for Kowal (1985) these studies were based on certain unbalanced designs chosen as a subjective manner and, therefore, fail to provide an overall coverage of other possible unbalanced situations. Except for Cox, Greenman and Fienberg (1985) which used a two-fold nested random model, all the aforementioned studies employed the one-way random model.

To study the effect of imbalance on a quantity of interest, a method for generating designs having a specific degree of imbalance is needed. Three such methods were introduced. Adams and Searles (1961) considered the one-way random model and used one of two manners of imbalance proposed by Adams and Farrow (1941) to compare designs. Adams and Searles measured the relative efficiency of ANOVA estimators with respect to MNQU estimators in terms of the ratio of variances of these

excesses. They generated designs by simply manipulating several combinations of sample size ranging from the best to the most unbalanced situations for a given total number of observations (N) and a given number of groups (k) with a fixed first group sample size (n_1). They were thus able to compare designs with different degrees of imbalance after fixing N, k , and n_1 . However, they were unable to compare “equally unbalanced” designs, i.e., designs with the same degree of imbalance for fixed N and k .

Donner and Krewt (1987) developed a method for generating designs, using the one-way random model, based on an underlying probability distribution, such as the truncated negative binomial distribution, presented below (see). They investigated the large sample relative efficiency (RE) of ANOVA estimators ($\hat{\mu}_1^2, \hat{\mu}_2^2$) with respect to the RE estimators ($\hat{\mu}_1^2, \hat{\mu}_2^2$) based on three replicates using Monte-Carlo simulation. Note that the equation for RE($\hat{\mu}_1^2, \hat{\mu}_2^2$) in Donner and Krewt (1987, formula 3.1, p. 104) was incorrectly cited from that of Chapp (1966, equation on p. 95). Examining the behavior of RE estimators was particularly interesting since there is no explicit form of these estimators. On the basis of a Monte-Carlo study, they reached the conclusion that the effect of imbalance on RE($\hat{\mu}_1^2, \hat{\mu}_2^2$) depends on the choice of the intraclass correlation coefficient ($\rho = \frac{\sigma_b^2}{\sigma_b^2 + \sigma_w^2}$). As a measure of imbalance, they used one of Horvitz and Fienberg measures, namely $\rho = \sigma(D)$. One of the main deficiencies of their method is that the generated design does not correspond exactly to a specified degree of imbalance; rather, it has on the average that degree.

Krewt (1990) proposed a new method for constructing designs for the one-way model having a specified degree of imbalance. More specifically, given a measure of imbalance, $n_{1,0} = \sigma(D)$, a design is to be generated having a specified value of ρ and a specified value of $N (= \sum_{i=1}^k n_i)$ for a given k . That the design $D = \{n_1, \dots, n_k\}$ should satisfy $\sum_{i=1}^k n_i = N$ and $\sum_{i=1}^k n_i^2 = \frac{N^2}{k}$. By letting $n_i = \frac{N}{k}, 0 < n_i < 1$, the set

of all feasible solutions consists of the coordinates of the points in the interior of a regular $(k-1)$ -dimensional simplex that belong to the intersection of a $(k-1)$ -dimensional hyperplane and a $(k-1)$ -dimensional hypersphere. This intersection is a $(k-2)$ -dimensional hypersphere with a radius $\left[\frac{1}{k!}\right]^{1/2}$ centered at the centroid of the simplex. Since a set of feasible solutions is a subset of this hypersphere, the values of n_i having a specified value of N were obtained using spherical coordinates associated with the hypersphere. Once values of $n_1, n_2, \dots, n_k$ were determined, if Nn_i is a positive integer for $i = 1, 2, \dots, k$, then $n_i = Nn_i$, $i = 1, 2, \dots, k$, produce a design having the specified values of N and q . Otherwise, the equations $\sum_{i=1}^k n_i = N$ and $\sum_{i=1}^k n_i^2 = \frac{N^2}{k}$ do not have an exact integer solution. Kiani showed how to obtain an approximate solution to these equations in an optimal manner. The proposed design would be $\{n_1, \dots, n_k\}$.

As a quantity of interest, Kiani (1986) used a scaled version of the ANOVA estimator of σ_e^2 for the one-way model, namely, the quantity $\mu = \frac{1}{2} \text{Var}[d_e^2]$ in order to determine the effect of data imbalance on the precision of estimating σ_e^2 . Values of μ were obtained using designs generated with specified combinations of N , k , and p values. An empirical relationship was established between μ , on one hand, and N , k , and p on the other hand. This was done by fitting a response surface model in which μ is the response variable and N , k , and p are the control variables. By exploring the response surface corresponding to the empirical relationship over the factor space associated with the specified combinations of N , k and p it is possible to assess the effects of these parameters on the precision of estimating σ_e^2 .

There are at least three other papers that address the effect of imbalance on the power of tests. Denner and Kari (1987) investigated the effect of imbalance on the power of the F -test for the variance components in the one-way random model. Using the technique proposed by Denner and Kari (1987) to generate designs for a

given degree of imbalance, they compared the expected power of the usual F -test of $H_0: \sigma^2_0 = 0$ to that of the corresponding likelihood ratio(LR)-test for different degrees of imbalance. They found that the LR-test could be appreciably more powerful than the F -test for extremely unbalanced designs.

Bloch (1981) studied the effects of heteroscedasticity and data imbalance on the power of the usual F -test of $H_0: \sigma^2_0 = 0$ when the error variances in the one-way random model are heterogeneous. He found that heteroscedasticity increases the power of the test in situations where the larger variances are associated with the smaller group sizes. If the larger variances are associated with the larger group sizes, the power is decreased. A small increase in the power, due to heteroscedasticity, was noted with balanced data.

El Khatami and Serfling (1984) compared designs on the basis of power of tests. They proposed the modified harmonic mean test procedure for variance components and compared this test with the locally most powerful (LMP)-test and with the Wald test for the one-way random model. They considered ratios of expected powers of these tests to the power of the most powerful (MP) test at $\alpha = 1, 0.5$, and 0.1 , where $\alpha = \frac{1}{N}$. They argued that, when there is no a priori knowledge of α (or σ^2_0), or when the design is badly unbalanced, the modified harmonic mean test is recommended. Since power is a function of α and the design (namely a, b and N for the one-way classification model), the effects of design and α on the powers of the tests should be examined. We can possibly accomplish this by establishing an empirical relationship between the power of a test and N, a , and α using Khuri's (1980) approach.

Thomas and Hubertson (1986) studied interval estimation of variance-components using the one-way random model. They considered a statistic which is a function of the harmonic mean of the s_i^2 's and of the sample variance of the treatment means. They argued that the exact distribution of this statistic can be approximated by a chi squared distribution. The adequacy of this approximation depends on the design

and the variance components. To verify this, for a given design and values of variance components, the coverage probability of a confidence interval for variance components was obtained and compared with the nominal value of the confidence coefficient, $1-\alpha$. Here also, we can investigate how this approximation works more fully using Chen's [2004] approach.

CHAPTER 3 DESIGN COMPARISONS

3.1 Introduction

The quality of estimation of variance components depends to a large extent on the design used to generate the response data. The majority of the published work in this design area appeared in the 1960's and 1970's [see the review articles by Anderson (1975) and Khan and Huda (1985) for a comprehensive coverage of the relevant literature]. For the most part, the criteria used in the selection of designs were based on single-valued functions such as the trace or determinant of the variance-covariance matrix of the vector of variance component estimators [see, for example, Goldsack and Gupta (1974)].

An underlying feature of designs for variance components estimation is their dependency on the values of the variance components themselves. Thus the choice of an efficient design cannot be made without some knowledge of these parameters especially when a single-valued criterion is used. The issue applies to the comparison of several designs. This dependency problem was noted by Sena, Challa, and McCulloch (1968, p. 70) in their study of the effect of imbalance on the variance of the ANOVA estimator of σ_B^2 , the among-groups variance component for an one-way random model. Loh (1983) concluded that the dependency of a criterion function on the unknown parameters makes the design problem a nonlinear one. To resolve this problem, Anderson (1981), Mukerjee and Huda (1983), and Deyoung and Sebastian (1985) obtained locally optimal designs in the sense of Chernoff (1953). This approach requires the specification of some prior knowledge of the variance components. As a

result, the efficiency of the generated designs will be dependent on the reliability of the postulated prior knowledge. Another approach to the design dependency problem is based on the use of the Bayesian methodology, which regards variance components as random variables. The latter approach requires the specification of a prior distribution for the variance components. Obviously, different prior distributions can result in different designs. Estimates are obtained from posterior distribution of the variance components.

Another factor that affects the choice of design is the method used to estimate the model's variance components. Muir and Anderson (1979) compared designs using maximum likelihood (ML) for the estimation of variance components in a two-way cross classification random model. Goldsmith and Taylor (1979) used analysis of variance (ANOVA) estimation to compare designs for a two-fold nested random model.

In this chapter, a graphical approach is presented for comparing designs for estimating variance components as well as for comparing different methods of estimation with a particular design. The proposed approach uses the so-called quantile dispersion graphs (QDGs) that were first introduced in Kiani (1997) in connection with the balanced two-way random model. The QDGs consist of plots of the maxima and minima, over some region in the parameter space, of the quantiles of a variance component estimator. An approach using the so-called empirical quantile dispersion graphs (EQDGs) is proposed in this study when exact quantile values are unavailable. The EQDGs are obtained by plotting the maxima and minima of the empirical quantiles of a variance component estimator. These plots provide a comprehensive picture of the quality of estimation within particular designs. They can also be used to compare designs for the same model. Two methods of estimation will be considered, namely, ANOVA and ML.

In Section 3.2, designs for one-way random model are considered. Two-way cross classification random model is investigated in Section 3.3. Section 3.4 is devoted for threshold mixed random model and is followed by comments in Section 3.5.

3.2. The One-Way Random Model

3.2.1. Introduction

Consider the unbalanced one-way random model,

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad (3.1)$$

$i = 1, 2, \dots, J$, $j = 1, 2, \dots, n_i$ where μ is a fixed unknown parameter, α_i is the effect of level i of the treatment factor and ε_{ij} is a random error. We assume that α_i and ε_{ij} are independently distributed such that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$, $i = 1, 2, \dots, J$, and $j = 1, 2, \dots, n_i$. The model is balanced if the n_i 's are equal that is $n_1 = n_2 = \dots = n_J = n$. Model (3.1) can be written in vector form as

$$\mathbf{y} = \mu \mathbf{1}_N + [\sigma_\alpha^2 \mathbf{I}_N] \boldsymbol{\alpha} + \boldsymbol{\varepsilon}, \quad (3.2)$$

where $N = \sum_{i=1}^J n_i$, $\mathbf{y} = (y_{11}, y_{12}, \dots, y_{1n_1}, \dots, y_{J1}, y_{J2}, \dots, y_{Jn_J})'$, $\mathbf{1}_N$ is the vector of ones of order $N \times 1$, $\boldsymbol{\alpha}$ is the symbol of direct sum of matrices, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_J)'$, and $\boldsymbol{\varepsilon}$ is defined analogously to $\mathbf{y}$. The mean of $\mathbf{y}$ is $\mu \mathbf{1}_N$, and the variance covariance matrix (denoted by $\mathbf{E}$) is of the form

$$\mathbf{E} = \sigma_\alpha^2 \sigma_\varepsilon^2 \mathbf{J}_N + \sigma_\varepsilon^2 \mathbf{I}_N, \quad (3.3)$$

where $\mathbf{J}_N$ is the identity matrix of order $N \times N$ and $\mathbf{J}_{n_i}$ is the matrix of ones of order $n_i \times n_i$.

The proposed graphical technique for computing designs for model (3.1) uses quantities of the distribution of $\hat{\mu}$, where σ_α^2 denotes an estimator of σ_α^2 . ANOVA estimation is used in Sections 3.2.2 and 3.2.3 while ML estimation is considered in Section 3.2.4. Comparisons of these estimation methods for given designs are made

in Section 3.1.5. In Section 3.1.6, designs for estimating the ratio $\rho = \frac{\sigma_{\beta}^2}{\sigma_{\epsilon}^2 + \sigma_{\beta}^2}$ are compared, using the ANOVA and ML estimates.

3.1.3. Overview of $\hat{\Sigma}_\beta$ using ANOVA estimation

The ANOVA table for model (3.1) is of the form

Source	d.f.	Sum of Squares	Expected Mean Squares
Among groups	$k - 1$	SS_A	$\sigma_{\epsilon}^2 + \sigma_{\beta}^2$
Within groups	$N - k$	SS_B	σ_{ϵ}^2
Total(adjusted)	$N - 1$	SS_T	

where

$$d = \frac{1}{k-1} \left(N - \frac{1}{N} \sum_{i=1}^k n_i^2 \right) \quad (3.4)$$

$$SS_A = \mathbf{y}' \mathbf{Q}_A \mathbf{y}, \quad (3.5)$$

$$SS_B = \mathbf{y}' \mathbf{P}_B \mathbf{y}, \quad (3.6)$$

and

$$\mathbf{Q} = \sigma_{\epsilon}^2 \left(\frac{1}{n_k} \mathbf{J}_{n_k} \right) - \frac{1}{N} \mathbf{J}_N, \quad (3.7)$$

$$\mathbf{P} = \mathbf{I}_N - \sigma_{\epsilon}^2 \left(\frac{1}{n_k} \mathbf{J}_{n_k} \right). \quad (3.8)$$

The matrix $\mathbf{Q}\mathbf{E}$ can be expressed as (see Kiani, Mathew and Sader 1998, p. 94)

$$\mathbf{Q}\mathbf{E} = \left[\sigma_{\epsilon}^2 n_k \mathbf{J}_{n_k} - \frac{1}{N} \mathbf{I}_{n_k} \mathbf{d} \right] \mathbf{e}_k^T + \left[\sigma_{\epsilon}^2 \left(\frac{1}{n_k} \mathbf{J}_{n_k} - \frac{1}{N} \mathbf{J}_N \right) \right] \mathbf{e}_k^T, \quad (3.9)$$

where $\mathbf{d} = (n_1 \mathbf{1}_{n_1}^T, n_2 \mathbf{1}_{n_2}^T, \dots, n_k \mathbf{1}_{n_k}^T)$. In general, the matrix is not a multiple of an idempotent matrix unless the design is balanced. Hence, SS_A is not distributed as

a scalar multiple of a chi-squared variate as in the case with balanced models (see, for example, Seasholtz, Casella, and McCulloch, 1988, Theorem 3.2, Appendix B.4, p. 187). The following theorem shows that SS_{ε} can be expressed as a linear combination of independent central chi-squared variates.

Theorem 3.2.1 For a p -dimensional vector $\mathbf{y} \sim N(\boldsymbol{\mu}, \Sigma)$ and a symmetric matrix $\mathbf{A}$, $\mathbf{y}'\mathbf{A}\mathbf{y}$ can be expressed as a linear combination of independent central chi-squared variates of the form

$$\mathbf{y}'\mathbf{A}\mathbf{y} \stackrel{d}{=} \sum_{i=1}^p \lambda_i \chi_{\lambda_i}^2$$

where $\lambda_1, \lambda_2, \dots, \lambda_p$ are the distinct nonzero eigenvalues of $\mathbf{A}\Sigma$ with multiplicities $\nu_1, \nu_2, \dots, \nu_p$ respectively.

Proof [See Johnson and Kotz, 1969, Ch. 19, pp. 160–165]. □

Corollary 3.2.1 The result of Theorem 3.2.1 remains true if $\mathbf{y} \sim N(\boldsymbol{\mu}, \Sigma)$ provided that $\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} = 0$ and $\mathbf{A}$ is symmetric positive semidefinite.

Proof Assuming $\mathbf{y} \sim N(\boldsymbol{\mu}, \Sigma)$, let $\mathbf{y} = \Sigma^{-1/2}\mathbf{u} \sim N(\Sigma^{-1/2}\boldsymbol{\mu}, \mathbf{I}_p)$. Then, by the spectral decomposition theorem

$$\begin{aligned} \mathbf{y}'\mathbf{A}\mathbf{y} &= \mathbf{y}'\Sigma^{-1/2}\Sigma^{1/2}\mathbf{A}\Sigma^{-1/2}\mathbf{y} \\ &= \mathbf{y}'\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}'\mathbf{y} \end{aligned}$$

where $\boldsymbol{\Lambda} = \text{diag}(\lambda_i)$, $i = 1, 2, \dots, N$, with λ_i being the i^{th} eigenvalue of $\Sigma^{-1/2}\mathbf{A}\Sigma^{-1/2}$, and $\mathbf{P}$ is an orthogonal matrix whose columns are orthonormal eigenvectors of $\Sigma^{-1/2}\mathbf{A}\Sigma^{-1/2}$ corresponding to the λ_i 's. Let $\mathbf{w} = \mathbf{P}'\mathbf{y}$, then $\mathbf{w} \sim N(\mathbf{P}'\Sigma^{-1/2}\boldsymbol{\mu}, \mathbf{I})$ and

$$\begin{aligned} \mathbf{y}'\mathbf{A}\mathbf{y} &= \mathbf{w}'\mathbf{w} \\ &= \sum_{i=1}^p \lambda_i w_i^2. \end{aligned}$$

where $\lambda_1, \lambda_2, \dots, \lambda_r$ are the distinct nonzero eigenvalues of $\mathbf{B}^T \mathbf{A} \mathbf{B}^T$, and hence of $\mathbf{A} \mathbf{B}$ with multiplicities $\alpha_1, \alpha_2, \dots, \alpha_r$, respectively. Also, $\mathbf{W}_i^T$ is distributed as $\chi_{\alpha_i}^2(\mathbf{B}_i)$, $\mathbf{B}_i = \sum_{j=1}^p \mathbf{P}_j^T \mathbf{B}^{-1} \mathbf{a}_j (\mathbf{P}_j^T \mathbf{B}^{-1} \mathbf{a}_j)$ where $\mathbf{P}_i$ is a matrix of order $p \times \alpha_i$ whose columns are eigenvectors corresponding to λ_i . Thus $\mathbf{a}^T \mathbf{A} \mathbf{a} \stackrel{D}{=} \sum_{i=1}^r \lambda_i \chi_{\alpha_i}^2$ if and only if $\mathbf{B}_i = \mathbf{0}$ for all $i = 1, 2, \dots, r$ that is, $\mathbf{a}^T \mathbf{B}^{-1} \mathbf{P}_i \mathbf{P}_i^T \mathbf{B}^{-1} \mathbf{a} = \mathbf{0}$ for all $i = 1, 2, \dots, r$, or equivalently if and only if $\mathbf{a}^T \mathbf{B}^{-1} (\sum_{i=1}^r \lambda_i \mathbf{P}_i \mathbf{P}_i^T) \mathbf{B}^{-1} \mathbf{a} = \mathbf{0}$ (the sufficiency is due to $\lambda_i > 0$ which is true if $\mathbf{A}$ is positive semidefinite), that is, $\mathbf{a}^T \mathbf{B}^{-1} \mathbf{P} \mathbf{P}^T \mathbf{B}^{-1} \mathbf{a} = \mathbf{0}$, which is equivalent to $\mathbf{a}^T \mathbf{B}^{-1} \mathbf{B}^T \mathbf{A} \mathbf{B}^T \mathbf{B}^{-1} \mathbf{a} = \mathbf{0}$, that is, $\mathbf{a}^T \mathbf{A} \mathbf{a} = \mathbf{0}$. Hence, the result follows. $\square$

Let us now derive the ANOVA estimator of σ_e^2 by $\hat{\sigma}_{eA}^2$. Since the distribution of σ_{eA}^2 is scale dependent, we consider instead the estimator $\hat{W}_{eA} = \frac{\hat{\sigma}_{eA}^2}{\sigma_0^2}$ which is scale free. The distribution of W_{eA} can be obtained by using the following theorem.

Theorem 3.1.1 For the randomized one-way random model (3.1), the scaled ANOVA estimator of σ_e^2 , $W_{eA} = \frac{\hat{\sigma}_{eA}^2}{\sigma_0^2}$ is distributed as

$$W_{eA} \stackrel{D}{=} \frac{1}{d} \left[\frac{1}{d-1} \sum_{j=1}^d K_j \chi_{\alpha_j}^2 - \frac{1}{(N-d)\sigma_0^2} \chi_{\alpha_0}^2 \right], \quad (3.12)$$

$$\text{where } d = \frac{1}{d-1} \left[N - \frac{1}{N} \sum_{j=1}^d \alpha_j \right]$$

α_0 = number of distinct positive eigenvalues of $\mathbf{Q} \mathbf{Q}^T$

K_j = J^{th} positive eigenvalue of $\mathbf{Q} \mathbf{Q}^T$ with multiplicity m_j , $j = 1, 2, \dots, \alpha_0$

$$\sigma_0^2 = \frac{\sigma_e^2}{\sigma_0^2}$$

$$\mathbf{Q}^T = \mathbf{Q}_{eA}^T \mathbf{J}_{\alpha_0} + \frac{1}{q} \mathbf{I}_{\alpha_0}$$

and $\mathbf{Q}$ is the matrix defined in formula (3.7)

Proof. The expected value of the mean square of the between group effect and the error term are $E(MS_B) = d\sigma_1^2 + \sigma_2^2$ and $E(MS_E) = \sigma_2^2$, respectively, where $d = \mu_1^2(N - \frac{1}{2} \sum_{i=1}^d n_i^2)$. Hence, the ANOVA estimator of σ_1^2 is

$$\begin{aligned} s_{L_0}^2 &= \frac{1}{d}(MS_B - MS_E) \\ &= \frac{SS_B}{d(n-1)} - \frac{SS_E}{d(n-1)} \end{aligned}$$

Note that $\mathbf{Q}$ is a symmetric positive semidefinite matrix. Since $E(\mathbf{y}) = \mu\mathbf{1}_N$ and $\mathbf{Q}\mathbf{1}_N = \mathbf{0}$ by Corollary 3.2.1, $SS_B = \mathbf{y}^T \mathbf{Q} \mathbf{y} = \sum_{i=1}^d \lambda_i x_i^2$, where $\lambda_1, \lambda_2, \dots, \lambda_d$ are the distinct positive eigenvalues of $\mathbf{Q}\mathbf{Q}^T$ with multiplicities $m_1, m_2, \dots, m_d$, respectively, and $\mathbf{X}$ is defined as in equation (3.1). Let $\mathbf{X}^* = \frac{1}{d}\mathbf{X} = x_{L_0}^* \mathbf{I}_d + \frac{1}{d}\mathbf{L}$. Then

$$SS_B = d \sum_{i=1}^d \lambda_i x_{L_0,i}^2 \quad (3.12)$$

where $\lambda_1, \lambda_2, \dots, \lambda_d$ are the distinct positive eigenvalues of $\mathbf{Q}\mathbf{Q}^T$. On the other hand, the error sum of squares is distributed as

$$SS_E = \sigma_2^2 \chi_{n-d}^2. \quad (3.13)$$

Independently from SS_B . By combining equations (3.11) and (3.12), we find that $s_{L_0}^2$ is distributed as

$$s_{L_0}^2 \stackrel{d}{=} \frac{1}{d} \left(\frac{\sigma_1^2}{d-1} \sum_{i=1}^d \lambda_i x_{L_0,i}^2 - \frac{\sigma_2^2}{n-d} \chi_{n-d}^2 \right) \quad (3.14)$$

Hence, the theorem follows. $\square$

Corollary 3.3.1 For the balanced one-way random model, $y_{ij} = \mu + \alpha_i + \eta_{ij}$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, n$, the usual ANOVA estimator of σ_1^2 , $\hat{\sigma}_{L_0}^2 = \frac{SS_B}{d}$, is distributed as

$$\hat{\sigma}_{L_0}^2 \stackrel{d}{=} \frac{n\eta + 1}{(n-1)n\eta} \chi_{d-1}^2 - \frac{1}{n\eta(n-1)n} \chi_{n-d}^2, \text{ where } \eta = \frac{\sigma_1^2}{\sigma_2^2}. \quad (3.15)$$

Proof In this case, the expected values of mean squares of between group effect and the error effect are $E(MS_{\beta}) = m\sigma_{\beta}^2 + \sigma_{\epsilon}^2$ and $E(MS_{\epsilon}) = \sigma_{\epsilon}^2$. Therefore, the ANOVA estimate of σ_{β}^2 is given by $s_{\beta,a}^2 = \frac{1}{m}(MS_{\beta} - MS_{\epsilon})$. Since for balanced model, $SS_{\beta} = (m\sigma_{\beta}^2 + \sigma_{\epsilon}^2)\chi_{a-1}^2$, and if $\alpha \in (0, 1)$ is α ,

$$s_{\beta,a}^2 \leq \frac{(m\sigma_{\beta}^2 + \sigma_{\epsilon}^2)}{(a-1)} \chi_{a-1}^2 = \frac{\sigma_{\epsilon}^2}{m(a-1)} \chi_{a-1}^2 \quad (3.13)$$

Hence the result follows. $\square$

Corollary 3.13 The probability of a negative ANOVA estimate of σ_{β}^2 for the unbalanced one-way random model is

$$P(s_{\beta,a}^2 < 0) = P\left(\frac{1}{k-1} \sum_{j=1}^k S_j^2 \chi_{k-1}^2 - \frac{1}{(n-k)(n-1)} \chi_{n-k-1}^2 < 0\right) \quad (3.14)$$

For the balanced one-way random model it is

$$P(s_{\beta,a}^2 < 0) = P\left(\frac{m+1}{(n-k)(n-1)} \chi_{k-1}^2 - \frac{1}{n(n-1)(n-1)} \chi_{n-k-1}^2 < 0\right) \quad (3.15)$$

Proof These formulas follow from (3.10) and (3.11) respectively. $\square$

The exact distributions of $W_{k,a}$ in (3.10) and (3.11) can be determined by its quantiles. The p th quantile of $W_{k,a}$, denoted by $q(p)$, is such that $P(W_{k,a} < q(p)) = p$. Since $W_{k,a}$ is a linear combination of independent chi-squared variates, an exact tabulation of its cumulative distribution function, $P(W_{k,a} < q(p))$, can be easily obtained using Davies' [1980] algorithm. The algorithm itself (denoted as *ANVAR*) is available via *XTNLIB*, an E-soft and the recently published internet system for statistical software. A tabulation of the quantiles $q(p)$ for given values of p is likewise possible through Davies' algorithm.

3.2.3. Comparison of designs using ANOVA estimation

The p^k quantile, q_k^p , of $\hat{W}_{1,11}$ in (3.18) depends on the unknown value of $\eta = \frac{\sigma_1^2}{\sigma_2^2}$ as well as on the design used by virtue of (3.18). By α -designs we mean a specification of the values of $n_1, n_2, \dots, n_k$ for a given k in the model of (3.1). Such a design is denoted by D . Thus, $D = \{n_1, n_2, \dots, n_k\}$. The dependency of the p^k quantile on η and D will henceforth be indicated by writing $q_k^p(\eta, D)$.

Kiefer (1967) introduced the notion of quantile dispersion graphs (QDGs) as a means to compare designs for estimating variance components on the basis of their quantile profiles. For model (3.1), the QDGs consist of plots of $Q_k^{p^k}(\eta)$ and $Q_k^{p^k}(\eta)$ against η where

$$Q_k^{p^k}(\eta) = \sup_{D \in \mathcal{D}_\eta} q_k^p(\eta, D), \quad (3.19)$$

$$Q_k^{p^k}(\eta) = \inf_{D \in \mathcal{D}_\eta} q_k^p(\eta, D) \quad (3.20)$$

Here $\mathcal{D}_\eta$ denotes a region in the parameter space of η . Such plots provide an overall pictorial assessment of the quality of ANOVA estimation for a given design D . Flat plots and low values of $Q_k^{p^k}(\eta) - Q_k^{p^k}(\eta)$ over the range of η indicate a high quality of estimation and are therefore desirable. This implies stability in the values of the ANOVA estimator of σ_{11}^2 and little sensitivity to changes in the values of η . In other words, one design is preferred over another if the QDGs of the former display smaller changes in the quantile values than those for the latter.

Let us consider the following two designs for model (3.1) with $k = 2$ and $N = 10$:

$$D_{1,2} = \{1, 1, 2, 11, 15\} \quad (3.26)$$

$$D_{2,2} = \{1, 1, 1, 1, 20\} \quad (3.27)$$

These designs were initially considered by Seneta, Casella and McCulloch (1988: 20-22-23) in their study of the effect of imbalance on the variance of $\hat{W}_{1,11}$. In addition to

$D_{1,1}$ and $D_{2,1}$, let us also consider the balanced design for $k = 1$ and $N = 20$, namely

$$D_0 = \{1, 1, 1, 1, 1, 1\} \quad (3.17)$$

The evaluation capabilities of these three designs, with regard to $\sigma_{1,1}^2$, will now be compared using the corresponding QEDs of $\hat{W}_{1,1}$ in (3.16) and (3.18).

For the balanced design D_0 by (3.14)

$$\begin{aligned} W_{1,1} &= \frac{\sigma_{1,1}^2}{\sigma_1^2} \\ E &= \frac{1}{8} \left[\frac{1}{3} \left(1 + \frac{1}{3} \right) \sigma_1^2 + \frac{1}{108} \sigma_0^2 \right] \\ &= \frac{39+1}{108} \sigma_1^2 + \frac{1}{108} \sigma_0^2. \end{aligned} \quad (3.18)$$

Applying now (3.16) in conjunction with $D_{1,1}$, we get

$$W_{1,1} \hat{E} \frac{1}{123} (K_{1,1}\sigma_1^2 + K_{2,1}\sigma_1^2 + K_{3,1}\sigma_1^2) + \frac{1}{108}\sigma_0^2, \quad (3.19)$$

where $K_{1,1}$, $K_{2,1}$, and $K_{3,1}$ are the distinct means eigenvalues of $\mathbf{QED}^*$ in Theorem 3.2.2 with multiplicities 5, 1, and 1 respectively. Similarly for the design $D_{2,1}$ we have

$$W_{1,1} \hat{E} \frac{1}{72} (K_{1,1}\sigma_1^2 + K_{2,1}\sigma_1^2) + \frac{1}{108}\sigma_0^2, \quad (3.20)$$

where, in this case, $K_{1,1}$ and $K_{2,1}$ are the distinct means eigenvalues of $\mathbf{QED}^*$ with multiplicities 1 and 1, respectively.

The quantiles of $\hat{W}_{1,1}$ for designs D_0 , $D_{1,1}$, and $D_{2,1}$ are now computed using formulas (3.16), (3.19), and (3.20), respectively in conjunction with Gamma' distribution as the random variate. For each design, values of $\mathbf{QED}^{(p)}(y)$ and $\mathbf{QED}^{(p)}(x)$ in (3.14) and (3.16) are obtained for each of several values of p ($0 \leq p \leq 1$). Then, the quantiles of $q_{1,1}(p, y)$ with respect to y is carried out over a region R_y of the parameter space given by

$$R_y = \{y \mid 0.1 \leq y \leq 0.6\} \quad (3.21)$$

More specifically, for a given p , $q_0(p, q)$ was computed using the following values of q : $q = 1.3, 2.5, 10.0$. The maximum and minimum of the resulting values of $q_0(p, q)$ provide approximate values of $Q_0^{max}(p)$ and $Q_0^{min}(p)$. The results are given in Table 3.1.

The QDCs for D_1 , D_{12} , and D_{22} are constructed on the basis of Table 3.1. A display of the QDCs for each pair of designs is given in Figure 3.1. The following observations can be made:

- The variability in the quantile values with respect to q for D_1 and D_{12} are quite similar [see Figure 3.1(a)]. Design D_{22} , however, displays a larger degree of variability than both D_1 and D_{12} [see Figure 3.1(b)]. This holds true for all values of p .
- There is a greater degree of uniformity in the quantile values for D_1 and D_{12} than for D_{22} . This gives stability in the values of $\sigma_{D_i}^2$ achieved with D_1 or D_{12} than with D_{22} .

On the basis of these observations it can be easily concluded that the estimation capability of D_{22} is better than that of D_{12} . The performance of D_1 is surprisingly similar to that of D_2 with the latter being slightly better than the former [see Fig. 3.1(a)]. Thus the use of the QDCs made it quite easy to discriminate among the three designs, particularly between D_{22} and D_{12} . Such a task, however, is not always possible with other approaches. For example, using the value of $\frac{Var(\hat{\theta}_{D_i})}{\hat{\theta}_{D_i}^2}$ as a criterion for comparing designs, which Seneta, Casella, and McCulloch (1968, pp. 75-76) used, it is not possible to decide which of D_{12} and D_{22} is the better design uniformly over all values of η , so one has to refer Table 3.2. They noted that for some values of η , D_{22} is better than D_{12} because D_{22} has smaller values of $\frac{Var(\hat{\theta}_{D_i})}{\hat{\theta}_{D_i}^2}$ than D_{12} .

Table 3.4: Minimum and maximum possible values of $R_{L,2}$ for selected values of p for designs D_5 , D_{10} and D_{20}

p	D_5 (2, 3, 3, 1, 5)		D_{10} (1, 1, 1, 1, 10)		D_{20} (2, 1, 1, 1, 20)	
	min	max	min	max	min	max
0.05	-0.137	-0.056	-1.077	-0.023	-2.104	-0.007
0.25	-0.754	-0.144	-1.309	-0.090	-2.520	-0.080
0.45	-0.545	-0.054	-0.824	-0.115	-1.680	-0.170
0.55	-0.180	-0.009	-0.450	-0.009	-1.093	-0.014
0.65	0.160	-0.140	-0.113	-0.075	-0.771	-0.405
0.75	0.204	-0.091	-0.104	-0.070	-0.740	-0.530
0.85	0.050	0.020	-0.074	-0.014	-0.447	-0.705
0.95	1.011	1.007	-0.007	-0.004	-0.009	-1.010
0.99	1.026	1.000	1.114	1.011	1.000	-1.000
0.99	1.007	0.999	1.111	0.999	1.113	-0.999
0.99	1.000	0.997	0.999	0.999	0.999	-0.999
0.95	0.400	0.700	0.111	-0.507	-0.000	-0.000
0.99	0.000	0.070	0.107	-0.444	-0.000	0.000

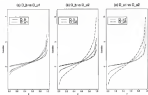


Figure 3.1: QDCs of $W_{n,1}$ for the three energies, D_0 , $D_{n,1}$ and $D_{n,2}$

Table 2: Values of $\frac{\text{Var}(\hat{\beta}_{\text{ML}})}{\text{Var}(\hat{\beta}_{\text{OLS}})}$ for designs D_0 , D_{10} , and D_{11}

Design	$\eta = \frac{\sigma^2}{\sigma_0^2}$				
	0.25	0.5	1	5	10
$D_0 = \{1, 2, 3, 5, 6\}$	0.1953	0.3439	0.7160	2.4247	33.0262
$D_{10} = \{1, 2, 3, 10, 11\}$	0.1793	0.4436	1.4116	6.0640	111.0260
$D_{11} = \{1, 2, 3, 1, 11\}$	0.2736	0.6030	1.5579	6.4673	61.3596

However, for other values of η ($\eta > 1$), D_{11} appears to be worse than D_{10} . They indicated that “The value of η at which $\text{Var}(\frac{\hat{\beta}_{\text{ML}}}{\hat{\beta}_{\text{OLS}}})$ changes from increasing to decreasing as redundancy increases is, of course, not the same for all situations, but depends in no simple way on N , k , and the α -value” (Sacks, Goel, and McCulloch, 1981, p. 74). However, based on the Figure 2.1, we can easily see the preference of design D_{11} to design D_{10} and the redundancy between design D_0 and D_{11} .

2.2.4 Comparison of designs using ML estimation

In this section, we consider the problem of design comparison for model (3.1) using maximum likelihood (ML) estimation of σ_0^2 . Let $\hat{\sigma}_{\text{ML}}^2$ denote the ML estimate of σ_0^2 and let $\hat{W}_{\text{ML}} = \frac{\hat{\sigma}_{\text{ML}}^2}{\sigma_0^2}$. Unlike the case of ANOVA estimation in Section 2.2.2, $\hat{\sigma}_{\text{ML}}^2$ does not have a closed form expression when the data set is split-balanced. The estimate $\hat{\sigma}_{\text{ML}}^2$ can only be computed numerically using, for example, EM or MIXED procedure in R or JAGG. Furthermore, the distribution of $\hat{\sigma}_{\text{ML}}^2$ is known only asymptotically. Consequently, the quantiles, and hence the QEDs for $\hat{W}_{\text{ML}}$, cannot be obtained exactly. They can, however, be obtained approximately using the so-called empirical quantile dispersion graph (EQDG) for $\hat{W}_{\text{ML}}$. This can be developed through Monte Carlo simulation. An outline of this process follows:

- (a) Without any loss of generality we can assume that $\mu = 0$ in model (2.1). We then have

$$y_j = \alpha_j + \varepsilon_j, \quad j = 1, 2, \dots, k, \quad j = 1, 2, \dots, N. \quad (3.17)$$

We recall that $\eta = \begin{pmatrix} \alpha \\ \varepsilon \end{pmatrix}$. Hence, $\alpha_j \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_k)$ and $\varepsilon_j \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_N)$. Let $\mathcal{D}$ denote a region in the parameter space of the pair $\{\eta, \sigma^2\}$.

- (b) For a given $\{\eta, \sigma^2\}$ in $\mathcal{D}$ and a given design $D = \{n_1, n_2, \dots, n_k\}$ a random vector $\mathbf{y}$ is generated on the basis of model (3.17) – our random numbers, which we denote by $\mathbf{y}_k$, is generated from $N(\mathbf{0}, \sigma^2 \mathbf{I}_N)$. In addition, n_k independent random numbers, denoted by $\varepsilon_{k1}, \varepsilon_{k2}, \dots, \varepsilon_{kn_k}$, are generated from $N(0, \sigma^2)$ independently of $\mathbf{y}_k$. This process produces the vector $[y_1 + \varepsilon_{11}, y_1 + \varepsilon_{12}, \dots, y_1 + \varepsilon_{1n_1}]^T$ which represents a sample of response values from the i^{th} group ($i = 1, 2, \dots, k$). By repeating the process independently k times and combining the results, we obtain a vector $\mathbf{y}$ consisting of $\sum_{i=1}^k n_i = N$ observations.

- (c) Using the response vector $\mathbf{y}$ obtained in step (b), a ML estimate, $\hat{\sigma}_{ML}^2$, of σ^2 is computed using, for example, EMML (EMML) procedure in EML (EMML). A value of $\hat{\mathbb{W}}_{ML} = \frac{1}{\hat{\sigma}_{ML}^2}$ is then obtained.

- (d) By repeating steps (b) and (c) a sufficient number of times, we can obtain repeated quantiles of $\hat{\mathbb{W}}_{ML}$ for selected values of p for a given $\{\eta, \sigma^2\}$. We denote the p^{th} empirical quantile thus obtained by $\hat{q}_p(\mathbf{y}, \eta, \sigma^2)$.

- (e) Steps (b)–(d) are repeated several times corresponding to different values of $\{\eta, \sigma^2\}$ selected from the region $\mathcal{D}$. The set of such selected point is denoted by $\mathcal{T}$. Thus $\mathcal{T} \subset \mathcal{D}$.

(f) The maximum and minimum of the y^2 required quantile values over the set T in step (e) are obtained. We then have

$$\hat{Q}_0^{(y^2)}(p) = \max_{(y, y^2) \in T} \text{val}(p, y, y^2) \quad (3.26)$$

$$\hat{Q}_2^{(y^2)}(p) = \min_{(y, y^2) \in T} \text{val}(p, y, y^2) \quad (3.27)$$

(g) Values of $\hat{Q}_0^{(y^2)}(p)$ and $\hat{Q}_2^{(y^2)}(p)$ are plotted against p to obtain the QQQs for W_{y^2} .

Let us now illustrate the application of the above procedure using the same designs, D_0 , $D_{0.5}$ and $D_{0.8}$, as in Section 3.1.3. The region V is chosen as

$$V = \{(y, y^2) : 0.2 \leq y \leq 0.8, \quad 0.2 \leq y^2 \leq 0.6\} \quad (3.28)$$

The set T mentioned in step (e) consists of 500 points selected from V such that

$$T = \{(y, y^2) : y = 0.2(0.2)0.8, \quad y^2 = 0.2(0.2)0.6\}$$

A SAS Macro (SAS 1988) was written to compute $\text{val}(p, y, y^2)$. A total of 1000 replications in step (a) were made to calculate values of $\hat{Q}_0(p, y, y^2)$ for each design and each (y, y^2) in the region T . Table 3.2 gives values of $\hat{Q}_0^{(y^2)}(p)$ and $\hat{Q}_2^{(y^2)}(p)$ at 0.25 and 0.75 respectively for D_0 , $D_{0.5}$ and $D_{0.8}$. Plots of these values for each pair of designs are given in Figure 3.2. As in Figure 3.1, we note here that the variability in the quantile values for D_0 and $D_{0.5}$ are similar [see Figure 3.2(a)]. However, the superiority of $D_{0.5}$ to $D_{0.8}$ is not as evident as in Figure 3.1 [see Figure 3.2(c)].

3.3.3. Comparison of ANOVA and ML estimation

We can use the information obtained in Sections 3.3.1 and 3.3.4 to compare ANOVA estimation with that of ML with regard to σ_y^2 in a limited sense since the regions S_0 and V in (3.16) and (3.28), respectively, are not the same. The QQQs

Table 3.5: Minimum and maximum expected quantile values of W_{test} for selected values of p for designs $\mathcal{D}_0$, $\mathcal{D}_{0.1}$ and $\mathcal{D}_{0.2}$

p	$\mathcal{D}_0$ [1, 3, 4, 5, 6]		$\mathcal{D}_{0.1}$ [1, 1, 3, 4, 5, 6]		$\mathcal{D}_{0.2}$ [1, 1, 1, 3, 4, 5]	
	min	max	min	max	min	max
0.001	0.000	-0.139	0.000	-0.009	0.000	-0.000
0.01	0.000	-0.797	0.000	-0.259	0.000	-0.001
0.10	0.000	-0.919	0.000	-0.585	0.000	-0.001
0.20	0.000	-0.684	0.000	-0.408	0.000	-0.007
0.30	0.127	0.000	0.000	-0.093	0.000	-0.007
0.40	0.411	0.797	0.000	0.747	0.000	0.711
0.50	0.695	0.919	0.004	0.610	0.000	0.670
0.60	0.979	1.138	0.001	1.081	0.000	1.061
0.70	1.107	1.000	1.100	1.009	1.107	1.000
0.80	1.000	0.900	1.000	0.901	1.000	0.910
0.90	0.700	0.777	1.000	0.970	0.704	0.971
0.95	0.300	0.500	0.904	0.900	0.000	0.707
0.99	0.000	0.200	0.000	0.000	0.000	0.000

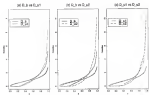


Figure 3.3: EVPIs of W_{opt} for the three designs, D_0 , D_1 , and D_2 .

for $W_{L,1}$ and the EQOCs for $W_{L,2}$ are shown together for each design in Figure 3.1. For the balanced design D_0 , the graphs for ANOVA and ML are almost identical, except for the possibility of getting negative values with the ANOVA estimate [see Figure 3.1(a)]. Note that for balanced data, restricted maximum likelihood (REML) estimates are identical to the ANOVA estimates when the latter are nonnegative. For design D_0 , the similarity between ANOVA and ML values is moderate [see Fig. 3.1(b)]. The frequency of getting negative ANOVA estimates is greater with D_0 than with D_1 . Finally, for design D_2 , the distinction between ANOVA and ML is more pronounced than for the other two designs [see Figure 3.1(c)]. In this case, ML is preferred over ANOVA in terms of stability in the quantile values.

3.2.4 Estimation of the intraclass correlation coefficient

The intraclass correlation coefficient (ICC) for model (3.1) is defined as the ratio $\rho = \sigma_{\alpha}^2 / \sigma_{\alpha}^2 + \sigma_{\epsilon}^2$. This is a measure of association among the responses from the same treatment group. We now discuss the comparison of designs for estimating ρ .

The ANOVA estimate of ρ is $\hat{\rho}_A = \frac{\hat{\sigma}_{\alpha A}^2}{\hat{\sigma}_{\alpha A}^2 + \hat{\sigma}_{\epsilon A}^2}$, where $\hat{\sigma}_{\alpha A}^2$ and $\hat{\sigma}_{\epsilon A}^2$ are the ANOVA estimates of σ_{α}^2 and σ_{ϵ}^2 , respectively. Similarly, the ML estimate of ρ is $\hat{\rho}_M = \frac{\hat{\sigma}_{\alpha M}^2}{\hat{\sigma}_{\alpha M}^2 + \hat{\sigma}_{\epsilon M}^2}$, where $\hat{\sigma}_{\alpha M}^2$ and $\hat{\sigma}_{\epsilon M}^2$ are the ML estimates of σ_{α}^2 and σ_{ϵ}^2 , respectively. The quantiles of $\hat{\rho}_A$ can be obtained by using Davies' (1980) algorithm. If $q_p(\cdot)$ denotes the p^{th} quantile of ρ_A , then

$$P(\hat{\rho}_A < q_p(\cdot)) = P\{[1 - q_p(\cdot)\hat{\sigma}_{\alpha A}^2] - q_p(\cdot)\hat{\sigma}_{\epsilon A}^2 < 0\} \quad (3.11)$$

Since $\hat{\sigma}_{\alpha A}^2 = \frac{SS_A}{J-1}$ and $\hat{\sigma}_{\epsilon A}^2$ as given by (3.12), we have on the right-hand side of (3.11) a linear combination of independent chi-squared variables. Hence, Davies' algorithm applies. On the other hand, the quantiles of $\hat{\rho}_M$ can be obtained by simulation as in Section 3.2.4.

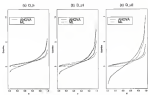


Figure 3.3: QCRs of R_{LSE} and expected values of W_{LSE} for the three designs D_0 , D_{opt} and D_{max} .

Table 3.4: Minimum and maximum quantile values of μ_x for selected values of p for designs D_0 , D_{21} , and D_{22}

p	D_0 (1, 1, 1, 1, 1)		D_{21} (1, 1, 1, 10, 11)		D_{22} (1, 1, 1, 1, 11)	
	min.	max.	min.	max.	min.	max.
0.01	-0.031	0.079	-0.007	0.104	-1.003	0.120
0.05	-0.106	0.016	-0.209	0.014	-0.703	0.012
0.10	-0.138	0.017	-0.140	0.043	-0.562	0.047
0.20	-0.040	0.000	-0.090	0.170	-0.340	0.303
0.30	-0.030	0.000	-0.000	0.370	-0.114	0.000
0.40	-0.071	0.077	0.105	0.441	-0.042	0.000
0.50	-0.120	0.000	0.100	0.475	-0.000	0.000
0.60	-0.170	0.012	0.100	0.600	0.147	0.010
0.70	-0.200	0.001	0.200	0.600	0.000	0.000
0.80	-0.121	0.000	0.320	0.600	0.000	0.040
0.90	-0.004	0.000	0.400	0.600	0.000	0.000
0.95	-0.100	0.000	0.000	0.071	0.010	0.000
0.99	-0.000	0.000	0.000	0.000	0.700	0.000

The minimum and maximum quantile values, using ANOVA estimation for p , for each of the three designs, D_0 , D_{21} , and D_{22} , are given in Table 3.4. The comparable results using ML estimation are shown in Table 3.5. The corresponding QQCs and EQQCs for each pair of designs are displayed in Figures 3.4 and 3.5, respectively. Here again, from Figure 3.4 we note the similarity in the QQC profiles for D_0 and D_{21} , and the superiority of both designs to D_{22} for ANOVA estimation. However, the EQQCs in Figure 3.5 are much more similar among designs for ML estimation.

Finally, the comparison between ANOVA and ML with regard to estimating p for each design is depicted in Figure 3.6 using QQCs for p_d and EQQCs for $p_{\hat{p}_d}$.

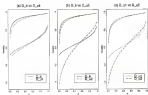


Figure 3.4: QOCs of J_n for the three designs, D_1 , D_2 , and D_3 .

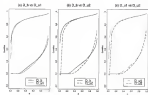


Figure 3.5: ROCs of J_0 for the three designs D_0 , $D_{0.5}$, and $D_{0.9}$.

respectively. We note that the two estimation methods do not exhibit any noticeable difference in the plots, except for the possibility of having negative $\hat{R}(\mathbf{D}_{22})$ estimates, especially with design $\mathbf{D}_{22}$.

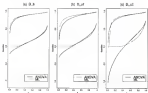


Figure 3-4: QQGs of p_1 and EQQGs of p_2 for the three designs, $\mathbf{D}_0$, $\mathbf{D}_{22}$, and $\mathbf{D}_{24}$

3.2 The Two-Way Cross Classification Random Model

3.2.1 Introduction

Consider the randomized two-way cross classification without interaction random model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk} \quad (3.21)$$

$i = 1, 2, \dots, I$, $j = 1, 2, \dots, J$, $k = 1, 2, \dots, K$, where μ is a fixed unknown parameter, α_i is the effect of the i^{th} level of the row factor A , β_j is the effect of the j^{th} level of the column factor B , and ε_{ijk} is a random experimental error. We assume that α_i and β_j are independently distributed such that $\alpha_i \sim N(0, \sigma_\alpha^2)$ and $\beta_j \sim N(0, \sigma_\beta^2)$, $i = 1, 2, \dots, I$, $j = 1, 2, \dots, J$, respectively. Moreover, ε_{ijk} is distributed as $N(0, \sigma_\varepsilon^2)$ independently from α_i and β_j , $k = 1, 2, \dots, K$. The model is balanced if the n_{ij} 's are equal, that is, $n_{11} = n_{12} = \dots = n_{IJ} = n$ (cf [6]). Model (3.21) can be written in vector form as

$$\mathbf{y} = \mu \mathbf{1}_N + \mathbf{X}_1 \boldsymbol{\alpha} + \mathbf{X}_2 \boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad (3.22)$$

where

$$\begin{aligned} \mathbf{y} &= \sum_{i=1}^I \sum_{j=1}^J \mathbf{y}_{ij}, \\ \mathbf{1}_N &= \text{the vector of ones of order } N = IJ, \\ \mathbf{X}_1 &= (\mathbf{x}_{11}^T, \mathbf{x}_{12}^T, \dots, \mathbf{x}_{1I}^T)^T, \text{ where } \mathbf{x}_i = \sum_{j=1}^J \mathbf{x}_{ij}, \\ \mathbf{X}_2 &= (\mathbf{x}_{11}^T, \mathbf{x}_{21}^T, \dots, \mathbf{x}_{J1}^T)^T, \\ \boldsymbol{\alpha} &= (\alpha_1, \alpha_2, \dots, \alpha_I)^T, \\ \boldsymbol{\beta} &= (\beta_1, \beta_2, \dots, \beta_J)^T, \end{aligned}$$

and σ is defined analogously to $\mathbf{g}$, as $N+1$ vector of observations. The mean of $\mathbf{g}$ is $\mu\mathbf{1}_N$, and its variance-covariance matrix, denoted by Σ , is of the form

$$\Sigma = \mathbf{X}_1\mathbf{X}_1'\sigma_1^2 + \mathbf{X}_2\mathbf{X}_2'\sigma_2^2 + \mathbf{I}_N\sigma_3^2 \quad (3.34)$$

Further, let us define the following matrices for later use

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{1}_N, \\ \mathbf{E}_2 &= [\mathbf{1}_N \quad \mathbf{X}_1], \\ \mathbf{E}_3 &= [\mathbf{1}_N \quad \mathbf{X}_1 \quad \mathbf{X}_2], \\ \mathbf{F}_1 &= \lambda\sigma(\hat{\lambda}_D)\lambda\sigma' \mathbf{1}_N, \\ \mathbf{F}_2 &= \mathbf{X}_2(\hat{\lambda}_D'\mathbf{X}_2')^{-1}\mathbf{X}_2', \text{ and} \\ \mathbf{F}_3 &= \mathbf{E}_2(\hat{\mathbf{E}}_2\hat{\mathbf{E}}_2')^{-1}\mathbf{E}_2', \end{aligned} \quad (3.35)$$

where $\hat{\mathbf{A}}$ is a generalized inverse of a matrix $\mathbf{A}$. Note that $\mathbf{F}_1 = \frac{1}{2}\mathbf{J}_N$, where $\mathbf{J}_N$ is the matrix of ones of order $N+1$.

In this section, we consider the question of the distributions of $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ for comparing designs as well as comparing two estimation methods of the variance components, namely, ANOVA and ML, where $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ denote estimators of σ_1^2 and σ_2^2 , respectively. These designs are chosen for comparison purposes. Those designs which were used by Hogg and Tanaka (1993), namely, D-16, BDB-16, and GDB-16, will be defined in detail in Section 3.2.2. ANOVA estimators are investigated in Section 3.2.3 and 3.2.4 while ML estimators are considered in Section 3.3.1. Comparisons of these estimation methods for each design are made in Section 3.2.5.

3.2.1. Question of $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ using ANOVA estimators

We shall use Type I sums of squares (SS) to measure the contribution of factors A and B . Type I SS are sometimes referred to as sequential sums of squares.

An ANOVA table for model (3.33) is

Source	d.f.	Type I Sum of Squares	Expected Mean Square
A	J_1	$SS_A = R(\mu, \alpha) - R(\mu, \alpha, \beta) - R(\mu)$	$k_1\sigma_\mu^2 + k_2\sigma_\alpha^2 + \sigma_\epsilon^2$
B	J_2	$SS_B = R(\mu, \alpha, \beta) - R(\mu, \alpha, \beta, \gamma) - R(\mu, \alpha)$	$k_3\sigma_\beta^2 + \sigma_\epsilon^2$
Error	J_3	$SS_E = \mathbf{y}'\mathbf{y} - R(\mu, \alpha, \beta)$	σ_ϵ^2
Total (adj.)	$N-1$	SS_T	

where J_1 , J_2 , and J_3 are the numbers of degrees of freedom for A, B, and the error term, respectively. More specifically, J_1 and J_2 are the numbers of linearly independent Type I estimable functions of the factors A and B, respectively. A function of parameters is defined as estimable if there is a linear function of the vector of observations $\mathbf{y}$ whose expected value is identical to the function of the parameters. Type I estimable functions are estimable functions based on the partitioning of Type I SS. For a more rigorous definition of an estimable function, see Graybill (1975, pp. 484-488). Also, $J_1 = B - J_2 - J_3 = 1$. The notation $R(\cdot)$ denotes a reduction in the sum of squares with the constraints of the functions (subsetting the model fixed) (see Sealed, 1971, pp. 198-200). For example, $R(\mu, \alpha, \beta)$ is the reduction in sum of squares for fitting model (3.22). Similarly, $R(\mu, \alpha)$ is the reduction in sum of squares for fitting the model $\mu_{ij} = \mu + \alpha_i + \alpha_{ij}$. Hence $R(\mu, \alpha, \beta) - R(\mu, \alpha)$ is the reduction in sum of squares due to fitting a model containing μ , an α -factor, and a β -factor having already fitted one with μ and an α -factor. This is a measure of the extent to which a model can explain more of the variation in μ_{ij} by including a column effect (β) in addition to an overall effect (μ) and a row effect (α). The coefficients of the variance components in the expected mean squares, namely, k_1 , k_2 , and k_3 , can be obtained by using, for example, the systematic method of Hurley and Rao (see Hurley, 1987).

The sums of squares SS_A , SS_B , and SS_C can be written as

$$SS_A = \mathbf{y}'\mathbf{Q}_A\mathbf{y}, \quad (3.36)$$

$$SS_B = \mathbf{y}'\mathbf{Q}_B\mathbf{y}, \quad (3.37)$$

$$SS_C = \mathbf{y}'\mathbf{K}\mathbf{y}, \quad (3.38)$$

where

$$\mathbf{Q}_A = \mathbf{P}_1 - \mathbf{P}_2, \quad (3.39)$$

$$\mathbf{Q}_B = \mathbf{P}_1 - \mathbf{P}_3, \quad (3.40)$$

$$\mathbf{K} = \mathbf{I}_n - \mathbf{P}_2, \quad (3.41)$$

and the matrices $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ are defined in (3.24). The sums of squares are denoted by MS_A , MS_B , and MS_C , respectively. The sums of squares SS_A and SS_B are not distributed as scalar multiple of a chi squared variate except for the balanced model case, with $\mathbf{X}$ in (3.24) the matrices $\mathbf{Q}_A\mathbf{X}$ and $\mathbf{Q}_B\mathbf{X}$ are not, in general, a multiple of an idempotent matrix unless the design is balanced. However, we can use Theorem 3.1.1 and Corollary 3.1.1 to express SS_A and SS_B as linear combinations of independent central chi-squared variates.

Let us denote Type I ANOVA estimators of σ_1^2 and σ_2^2 by s_{1A}^2 and s_{2A}^2 , respectively. Let us also denote the scaled version of these estimators by $\hat{W}_{1A} = \frac{s_{1A}^2}{\sigma_1^2}$ and $\hat{W}_{2A} = \frac{s_{2A}^2}{\sigma_2^2}$, respectively. The distributions of $\hat{W}_{1A}$ and $\hat{W}_{2A}$ can then be obtained by using the following theorem.

Theorem 3.1.1 For the unbalanced two-way cross classification without interaction random model (3.23) the scaled Type I ANOVA estimators of σ_1^2 and σ_2^2 , $\hat{W}_{1A}$ and

$\mathbf{E}_{\text{tot}}$, respectively are described as

$$\mathbf{E}_{\text{tot}} \triangleq \sum_{j=1}^J \gamma_j \mathbf{E}_j^{\text{tot}} \quad \text{and} \quad (2.43)$$

$$\mathbf{E}_{\text{tot}} \triangleq \sum_{j=1}^J \lambda_j \mathbf{E}_{\text{tot},j}^{\text{tot}} \quad (2.44)$$

where

J = number of distinct positive eigenvalues of $\mathbf{Q}_x \mathbf{E}_{\text{tot}}$

K = number of distinct positive eigenvalues of $\mathbf{Q}_x \mathbf{E}_p$

γ_j = j^{th} positive eigenvalue of $\mathbf{Q}_x \mathbf{E}_{\text{tot}}$ with multiplicity l_j , $j = 1, 2, \dots, J$

λ_j = j^{th} positive eigenvalue of $\mathbf{Q}_x \mathbf{E}_p$ with multiplicity m_j , $j = 1, 2, \dots, K$

$$\gamma_1 = \frac{\sigma_1^2}{2}$$

$$\gamma_2 = \frac{\sigma_2^2}{2}$$

$$\mathbf{Q}_x = \frac{1}{k_1 k_2} \mathbf{Q}_x + \frac{k_2}{k_1 k_1 k_2} \mathbf{Q}_x + \frac{(k_2 - k_1)}{k_1 k_1 k_2} \mathbf{R} \quad (3.45)$$

$$\mathbf{Q}_p = \frac{1}{k_2 k_1} \mathbf{Q}_p + \frac{1}{k_2 k_1} \mathbf{R} \quad (3.46)$$

$$\mathbf{E}_x = \frac{1}{\sigma_1^2} \mathbf{E} = \mathbf{X}_1 \mathbf{X}_1' + \mathbf{X}_2 \mathbf{X}_2' \left(\frac{\mathbf{E}}{\sigma_1^2} \right) + \mathbf{E}_p \left(\frac{1}{\sigma_1^2} \right), \quad \text{and} \quad (3.47)$$

$$\mathbf{E}_p = \frac{1}{\sigma_2^2} \mathbf{E} = \mathbf{X}_1 \mathbf{X}_1' \left(\frac{\mathbf{E}}{\sigma_2^2} \right) + \mathbf{X}_2 \mathbf{X}_2' + \mathbf{E}_p \left(\frac{1}{\sigma_2^2} \right) \quad (3.48)$$

The matrices $\mathbf{Q}_x$, $\mathbf{Q}_p$, and $\mathbf{R}$ are defined in (3.37), (3.40), and (3.44), respectively, and k_1 , k_2 , k_3 , k_4 , k_5 , and k_6 are defined in an APPENDIX table which is given earlier in this section.

Proof. From the ANOVA table, we have

$$\begin{aligned} s_{L_1}^2 &= \frac{1}{k_1}(M\bar{X}_1 - M\bar{X}_2) \\ &= \frac{1}{k_1k_2}SS_1 = \frac{1}{k_1k_2}SS_2 \\ &= \frac{1}{k} \left[\frac{1}{k_1k_2}Q_1 + \frac{1}{k_2k_1}Q_2 \right] \mathbf{y} \\ &= \frac{1}{k} Q_3 \mathbf{y} \end{aligned}$$

Since Q_3 is a symmetric positive semidefinite matrix, $E(\mathbf{y}) = \mu\mathbf{1}_N$ and $Q_3\mathbf{1}_N = \mathbf{0}$, by Corollary 3.2.1, $s_{L_1}^2 = \sum_{i=1}^r \lambda_i x_i^2$, where $\lambda_1, \lambda_2, \dots, \lambda_r$ are the distinct positive eigenvalues of $Q_3\mathbf{E}$ with multiplicities $m_1, m_2, \dots, m_r$, respectively, with $\mathbf{E}$ being defined as in (3.16). By letting $\Sigma_1 = \frac{1}{k}\mathbf{E} = \mathbf{X}_1\mathbf{X}_1' \left(\frac{1}{k} \right) + \mathbf{X}_2\mathbf{X}_2' \left(\frac{1}{k} \right)$, the distribution of $\mathbf{W}_{L_1} = \frac{s_{L_1}^2}{\sigma^2}$ is obtained.

The Type I ANOVA estimate of $s_{L_1}^2$ is

$$\begin{aligned} s_{L_1}^2 &= \frac{1}{k_1}(M\bar{X}_1 - \frac{k_2}{k_1}(M\bar{X}_2 - M\bar{X}_3) - M\bar{X}_3) \\ &= \frac{1}{k_1k_1}SS_1 - \frac{k_2}{k_1k_1k_2}SS_2 - \frac{(k_2 - k_1)}{k_1k_1k_2}SS_3 \\ &= \frac{1}{k} \left[\frac{1}{k_1k_1}Q_1 - \frac{k_2}{k_1k_1k_2}Q_2 - \frac{(k_2 - k_1)}{k_1k_1k_2}Q_3 \right] \mathbf{y} \\ &= \frac{1}{k} Q_4 \mathbf{y} \end{aligned}$$

Note that Q_4 is a symmetric positive semidefinite matrix and $Q_4\mathbf{1}_N = \mathbf{0}$. Again, by applying Corollary 3.2.1, $s_{L_1}^2 = \sum_{i=1}^r \gamma_i x_i^2$, where $\gamma_1, \gamma_2, \dots, \gamma_r$ are the distinct positive eigenvalues of $Q_4\mathbf{E}$ with multiplicities $\ell_1, \ell_2, \dots, \ell_r$, respectively. By defining $\Sigma_2 = \frac{1}{k}\mathbf{E} = \mathbf{X}_1\mathbf{X}_1' \left(\frac{1}{k} \right) + \mathbf{X}_2\mathbf{X}_2' \left(\frac{1}{k} \right) + \mathbf{X}_3\mathbf{X}_3' \left(\frac{1}{k} \right)$, the distribution of $\mathbf{W}_{L_2} = \frac{s_{L_2}^2}{\sigma^2}$ is obtained. $\square$

Corollary 3.3.1. For the balanced two-way cross classification without interaction random model $y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, n$, $W_{i..} = \sum_{j=1}^n y_{ij}^2$ and $V_{i..} = \sum_{j=1}^n y_{ij}$ are distributed as

$$W_{i..} \stackrel{d}{=} \frac{m_1 + 1}{(p-1)m_1} \chi_{i-1}^2 + \frac{1}{(p-1)(1-\frac{1}{p}m_1)} \chi_{(p-1)(p-1)}^2 \quad (3.44)$$

$$V_{i..} \stackrel{d}{=} \frac{m_2 + 1}{(p-1)m_2} \chi_{i-1}^2 + \frac{1}{(p-1)(1-\frac{1}{p}m_2)} \chi_{(p-1)(p-1)}^2 \quad (3.45)$$

where $m_1 = \frac{p}{p-1}$ and $m_2 = \frac{p}{p-1}$.

Proof. In this case, $E(W_{i..}) = m_1^2 + \sigma_\varepsilon^2$, $E(V_{i..}) = m_2^2 + \sigma_\varepsilon^2$, and $E(W_{i..}V_{i..}) = m_1^2$. Since, for the balanced model, $W_{i..} = (m_1^2 + \sigma_\varepsilon^2) \chi_{i-1}^2$, $V_{i..} = (m_2^2 + \sigma_\varepsilon^2) \chi_{i-1}^2$, $E(W_{i..}V_{i..}) = \sigma_\varepsilon^2 \chi_{(p-1)(p-1)}^2$ and $W_{i..}$, $V_{i..}$, and $E(W_{i..}V_{i..})$ are independently distributed, the ANOVA indicators of σ_ε^2 and σ_ε^2 are

$$\sigma_{i..}^2 = \frac{1}{p} W_{i..} - W_{i..}^2 \stackrel{d}{=} \frac{\sigma_\varepsilon^2 + m_1^2}{(p-1)m_1} \chi_{i-1}^2 + \frac{\sigma_\varepsilon^2}{(p-1)(1-\frac{1}{p}m_1)} \chi_{(p-1)(p-1)}^2 \text{ and}$$

$$\sigma_{i..}^2 = \frac{1}{p} V_{i..} - W_{i..}^2 \stackrel{d}{=} \frac{\sigma_\varepsilon^2 + m_2^2}{(p-1)m_2} \chi_{i-1}^2 + \frac{\sigma_\varepsilon^2}{(p-1)(1-\frac{1}{p}m_2)} \chi_{(p-1)(p-1)}^2$$

respectively. Hence the result follows. $\square$

The exact distribution of $W_{i..}$ in (3.44) and (3.45), and $V_{i..}$ in (3.44) and (3.45) can now be determined by these quantities. We can use Devore's (2006) algorithm to obtain exact values of quantiles.

3.3.1. Computation of desired alpha ANOVA indicators

The p^α quantile of $W_{i..}$, denoted by $w(p)$, is such that $P(W_{i..} \leq w(p)) = p$. For our model (3.44) this quantile depends on the unknown values of $m_1 = \frac{p}{p-1}$ and $m_2 = \frac{p}{p-1}$.

as well as on the design used by virtue of (3.45) and (3.46). The same applies for the p^{th} quartic of $W_{3,4}$ with (3.47) and (3.48). By a design we mean a specification of the values of $w_{11}, w_{22}, \dots, w_{nn}$ for a given r and s in the model of (3.32). We denote a design by $\mathcal{D}$. The dependency of the p^{th} quartic on n_1, n_2 , and $\mathcal{D}$ will therefore be indicated by writing $q_p^{\mathcal{D}}(p, n_1, n_2)$ for $W_{3,4}$ and $q_p^{\mathcal{D}}(p, n_1, n_2)$ for $W_{5,4}$.

Let us denote a region in the parameter space of n_1 and n_2 by $\mathcal{D}_{n_1, n_2}$. The QOCs for $W_{3,4}$ are obtained by plotting $Q_{3,4}^{\mathcal{D}}(p)$ and $Q_{3,4}^{\mathcal{D}}(p)$ against p , where

$$Q_{3,4}^{\mathcal{D}}(p) = \max_{(n_1, n_2) \in \mathcal{D}_{n_1, n_2}} q_3^{\mathcal{D}}(p, n_1, n_2) \text{ and} \quad (3.50)$$

$$Q_{3,4}^{\mathcal{D}}(p) = \min_{(n_1, n_2) \in \mathcal{D}_{n_1, n_2}} q_3^{\mathcal{D}}(p, n_1, n_2). \quad (3.51)$$

Similarly, the QOCs for $W_{5,4}$ are obtained by plotting $Q_{5,4}^{\mathcal{D}}(p)$ and $Q_{5,4}^{\mathcal{D}}(p)$ against p , where

$$Q_{5,4}^{\mathcal{D}}(p) = \max_{(n_1, n_2) \in \mathcal{D}_{n_1, n_2}} q_5^{\mathcal{D}}(p, n_1, n_2) \text{ and} \quad (3.52)$$

$$Q_{5,4}^{\mathcal{D}}(p) = \min_{(n_1, n_2) \in \mathcal{D}_{n_1, n_2}} q_5^{\mathcal{D}}(p, n_1, n_2). \quad (3.53)$$

Kiani and Anderson (1998) compared several designs for model (3.32). Among these designs, the following three designs are of particular interest.

- (I) the balanced design with $r = 2$, $s = 2$, and $N = 36$ which we denote by $\mathcal{D}_{22}$;
- (II) the so-called balanced incomplete block design with $r = 18$, $s = 36$ and $N = 36$ which we denote by $\mathcal{D}_{1836}$. The incidence matrix for this design is $\mathbf{X} = \mathbf{A}_2 \otimes \mathbf{J}_2$, where $\otimes$ is the symbol of Kronecker product of matrices, namely $\mathbf{A} \otimes \mathbf{B} = (a_{ij} \mathbf{B})$ where a_{ij} is the $(i, j)^{\text{th}}$ element of the matrix $\mathbf{A}$.

By an incidence matrix $\mathbf{X}$ we mean the $r \times s$ matrix having elements that are all 0 or 1. The locations of the 0's and 1's represent the incidence of items of

the model corresponding to the $\{x, y\}^{10}$ with $i = 1, 2, \dots, r, j = 1, 2, \dots, s$, and the data

- (E) the so-called *off-diagonal* design with $r = 14$, $s = 14$, and $N = 16$, which we denote by D_{14} . The incidence matrix for this design is $K = I_4 \otimes (I_7 - J_7)$.

Misner and Anderson (1971) compared these designs using asymptotic variance-covariance matrix of ML estimators of variance components. For example, they used the trace of the matrix, $\text{Var}(x_i^2)$ and $\text{Var}(x_j^2)$ where x_i^2 and x_j^2 are ML estimations of σ_i^2 and σ_j^2 , respectively. Their conclusions regarding these three designs, based on the trace criterion, can be summarized as follows (Misner and Anderson, 1971, p. 161)

- (a) If σ_i^2 is the dominant variance component, design D_{14} is preferred.
- (b) If $\sigma_i^2 \neq \sigma_j^2$ with either σ_i^2 or σ_j^2 larger than σ_j^2 , design D_{14} is preferred.
- (c) If both σ_i^2 and σ_j^2 are larger than σ_j^2 , design D_{14} is generally best.

Their conclusion of favoring certain unbalanced designs over the balanced design is quite interesting. However, we have, in general, little prior knowledge of variance components themselves which we would like to estimate. Therefore, their asymptotic results do not necessarily hold for small samples. Therefore, it is necessary to investigate the overall quality of estimation for the designs in a more comprehensive way. This can be accomplished by using QDCs.

The estimation capability of the designs, D_{14} , D_{16} and D_{28} , using ANOVA estimation of σ_i^2 and σ_j^2 can now be compared using the corresponding QDCs for W_{14} and W_{16} in (3.42) and (3.43), respectively.

The balanced design, D_9

The ANOVA table for model (1.12) with D_9 is

Source	d.f.	Sum of Squares	Expected Mean Squares
A	1	SS_A	$\sigma\alpha_1^2 + \sigma_1^2$
B	2	SS_B	$\sigma\alpha_2^2 + \sigma_2^2$
Error	15	SS_C	σ_1^2
Total(only)	18	SS_T	

The usual ANOVA estimates of σ_1^2 and σ_2^2 are by Corollary 3.3.1, distributed as

$$W_{1,11} \triangleq \frac{6\sigma_1 + 1}{10\sigma_1} x_1^2 \sim \frac{1}{10\sigma_1} x_1^2, \quad (3.16)$$

$$W_{2,11} \triangleq \frac{6\sigma_2 + 1}{10\sigma_2} x_2^2 \sim \frac{1}{10\sigma_2} x_2^2, \text{ respectively} \quad (3.17)$$

The balanced doublet, correlated design, $D_{1,11}$

The ANOVA table for model (3.15) using $D_{1,11}$, when Type I sums of squares are considered, is

Source	d.f.	Type I SS	Expected Mean Squares
A	1	$SS_A = \mathbf{g}^T \mathbf{Q}_A \mathbf{g}$	$\sigma\alpha_1^2 + \frac{1}{2}\sigma_1^2 + \sigma_1^2$
B	2	$SS_B = \mathbf{g}^T \mathbf{Q}_B \mathbf{g}$	$3\sigma_2^2 + \sigma_2^2$
Error	3	$SS_C = \mathbf{g}^T \mathbf{R}_B \mathbf{g}$	σ_1^2
Total(only)	6	SS_T	

Model (3.15) can be written as

$$\mathbf{g} = \mu\mathbf{1}_6 + (\mathbf{J}_2 \otimes \mathbf{I}_2)\alpha + (\mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2)\beta + \mathbf{e}$$

The variance-covariance matrix of $\mathbf{g}$, Σ in (3.16) is of the form

$$\Sigma = (\mathbf{J}_2 \otimes \mathbf{J}_2)\sigma_1^2 + (\mathbf{I}_2 \otimes \mathbf{J}_2 \otimes \mathbf{I}_2)\sigma_2^2 + \mathbf{I}_6\sigma_1^2,$$

and, hence, Σ_{β} in (3.45) and Σ_{β} in (3.47) are

$$\begin{aligned}\Sigma_{\beta_1} &= (I_{20} \oplus I_2) + (I_2 \oplus I_1 \oplus I_2) \left(\frac{10}{99} \right) + I_{20} \left(\frac{1}{99} \right) \\ \Sigma_{\beta_2} &= (I_{20} \oplus I_2) \left(\frac{10}{99} \right) + (I_2 \oplus I_1 \oplus I_2) + I_{20} \left(\frac{1}{99} \right),\end{aligned}$$

respectively. Also, $\hat{Q}_{\beta_1}$ in (3.46) and $\hat{Q}_{\beta_2}$ in (3.46) are

$$\begin{aligned}\hat{Q}_{\beta_1} &= \frac{1}{14}\hat{Q}_{\beta_1} - \frac{1}{153}\hat{Q}_{\beta_2} - \frac{1}{14}R, \text{ and} \\ \hat{Q}_{\beta_2} &= \frac{1}{14}\hat{Q}_{\beta_1} - \frac{1}{15}R\end{aligned}$$

respectively. Therefore, using the matrices $\hat{Q}_{\beta_1}$, $\hat{Q}_{\beta_2}$, Σ_{β_1} and Σ_{β_2} the distributions of $W_{1,2}$ and $W_{2,2}$ for the design B_{22} are obtained from Theorem 2.2.1

The off diagonal design, B_{22}

Similarly, for B_{22} , the ANOVA table is

Source	d.f	Type I SS	Expected Mean Squares
A	12	$SS_A = n \hat{Q}_{1,2}$	$\beta_1^2 + \frac{1}{2}\beta_2^2 + \sigma_e^2$
B	12	$SS_B = n \hat{Q}_{2,2}$	$\frac{1}{2}\beta_1^2 + \beta_2^2$
Error	4	$SS_C = n \hat{R}_2$	σ_e^2
Total(y.d.f.)	28	SS_T	

The model is

$$y = x\beta_{22} + (I_{20} \oplus I_2)\alpha + (I_2 \oplus I_1)\beta + \epsilon$$

where

$$\mathbf{K} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The variance-covariance matrix of $\mathbf{y}$ is

$$\Sigma = (\mathbf{I}_6 \oplus \mathbf{J}_2)\sigma_u^2 + (\mathbf{I}_6 \oplus \mathbf{K}\mathbf{K}')\sigma_v^2 + \mathbf{I}_6\sigma_e^2,$$

and hence Σ_{yy} and Σ_{yz} are

$$\Sigma_{yy} = (\mathbf{I}_6 \oplus \mathbf{J}_2) + (\mathbf{I}_6 \oplus \mathbf{K}\mathbf{K}') \left\{ \frac{19}{18} \right\} + \mathbf{I}_6 \left\{ \frac{1}{18} \right\},$$

$$\Sigma_{yz} = (\mathbf{I}_6 \oplus \mathbf{J}_2) \left\{ \frac{19}{18} \right\} + (\mathbf{I}_6 \oplus \mathbf{K}\mathbf{K}') + \mathbf{I}_6 \left\{ \frac{1}{18} \right\},$$

respectively. Also

$$\mathbf{Q}_u = \frac{1}{12}\mathbf{Q}_1 + \frac{4}{123}\mathbf{Q}_2 + \frac{19}{122}\mathbf{R},$$

$$\mathbf{Q}_v = \frac{1}{12}\mathbf{Q}_3 + \frac{1}{9}\mathbf{R}.$$

Using the matrices $\mathbf{Q}_u$, $\mathbf{Q}_v$, Σ_{yy} and Σ_{yz} , the distribution of W_{LS} and W_{PLS} for the design $\mathbf{D}_{LS}$ are also obtained from Theorem 2.2.1.

The quantiles of W_{LS} and W_{PLS} for designs $\mathbf{D}_{LS}$, $\mathbf{D}_{PLS}$, and $\mathbf{D}_{ML}$ are computed in comparison with Genstat (1995) algorithm. With each design, values of $Q_{LS}^{(p)}(y)$ and $Q_{PLS}^{(p)}(y)$ in (3.18) and (3.22), respectively, and values of $Q_{LS}^{(p)}(y)$ and $Q_{PLS}^{(p)}(y)$ in (3.19) and (3.23), respectively, are obtained for each of several values of p ($0 \leq p \leq 1$). The

Table 2.6: Minimum and maximum quartile values of $\hat{W}_{L,2}$ for selected values of p for designs D_{20} , D_{100} , and D_{1000}

p	D_{20}		D_{100}		D_{1000}	
	min.	max.	min.	max.	min.	max.
0.01	0.000	0.000	05.894	0.000	36.309	-0.000
0.05	0.040	0.017	10.009	-0.000	-0.000	-0.000
0.10	0.080	0.011	7.547	0.000	-0.000	-0.000
0.20	0.090	0.000	4.000	0.000	-0.000	-0.000
0.30	0.097	0.000	2.000	0.000	-0.000	-0.000
0.40	0.044	0.000	0.000	0.000	-0.000	-0.000
0.50	0.000	0.000	0.000	0.000	0.000	0.000
0.60	1.000	1.000	1.000	0.000	1.000	1.000
0.70	1.000	1.000	1.000	0.000	1.000	1.000
0.80	1.000	1.000	1.000	0.000	1.000	0.000
0.90	1.000	1.000	1.000	0.000	1.000	0.000
0.95	1.000	1.000	1.000	0.000	1.000	0.000
0.99	1.000	1.000	1.000	0.000	1.000	0.000

optimization of $q_{10}^L(p, \eta_1, \eta_2)$ and $q_{10}^U(p, \eta_1, \eta_2)$ with respect to (η_1, η_2) were carried out over a region $R_{(p, \eta_1, \eta_2)}$ of the parameter space given by

$$R_{(p, \eta_1, \eta_2)} = \{(p, \eta_1, \eta_2) \mid 0.5 \leq p_1 \leq 10.0, 0.5 \leq p_2 \leq 10.0\} \quad (2.34)$$

More specifically, for a given design and a given p , $q_{10}^L(p, \eta_1, \eta_2)$ is computed using the following values of (η_1, η_2) , $\eta_1 = \eta_2 = 0.5/0.02$ and $0.01/0.01$. The maximum and minimum of the resulting values of $q_{10}^L(p, \eta_1, \eta_2)$ for $W_{L,2}$ and $q_{10}^U(p, \eta_1, \eta_2)$ for $W_{U,2}$ which provide approximate values of $Q_{10}^{L,2}(p)$, $Q_{10}^{U,2}(p)$, and $Q_{10}^{L,2}(p)$, $Q_{10}^{U,2}(p)$, respectively, are given in Tables 2.6 and 2.7. The QQGs for the designs D_{20} , D_{100} , and D_{1000} are constructed on the basis of these values.

Table 5.7: Minimum and maximum possible values of H_{123} for selected values of y for designs $\mathcal{D}_{24}$, $\mathcal{D}_{242}$ and $\mathcal{D}_{243}$.

y	$\mathcal{D}_{24}$		$\mathcal{D}_{242}$		$\mathcal{D}_{243}$	
	min.	max.	min.	max.	min.	max.
0.05	-0.002	0.000	-1.023	0.101	-1.165	-0.163
0.06	-0.043	0.007	-0.970	0.207	-0.982	0.204
0.10	0.000	0.012	-0.633	0.405	-0.456	0.401
0.20	0.000	0.050	0.136	0.677	0.200	0.656
0.30	0.001	0.084	0.400	0.880	0.360	0.794
0.40	0.044	0.127	0.655	0.969	0.491	0.856
0.50	0.090	0.067	0.713	0.906	0.620	0.904
0.60	1.007	1.030	1.040	1.100	1.060	1.202
0.70	1.017	1.060	1.104	1.040	1.100	1.079
0.80	1.004	1.041	1.076	1.010	1.040	1.007
0.90	1.002	0.100	1.054	0.979	1.036	0.958
0.95	0.105	0.005	1.026	0.904	1.004	0.908
0.99	0.004	0.007	0.400	0.934	0.300	0.901

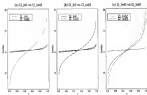


Figure 3.7 QDCs of $W_{1,2}$ for the beam designs, D_{20} , D_{200} , and D_{2000}

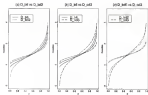


Figure 5.6: QDCs of $H_{2,2}$ for the three designs, D_{20} , D_{200} , and D_{2000} .

A display of the QEDs for $W_{1,1}$ and $W_{2,1}$ for each pair of designs are given in Figures 3.7 and 3.8, respectively. The following observations can be made from these figures.

- (a) The variability in the quantile values of $\hat{W}_{1,1} = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}$ with respect to (η_1, η_2) for design D_0 is much smaller than for designs $D_{0,0}$ and $D_{0,1}$ [see Figure 3.7(a,b)]. The variability for design $D_{0,0}$ is relatively smaller than for design $D_{0,1}$ [see Figure 3.8(a)]. Thus, we can get very stable Type-I ANOVA estimate of σ_1^2 with D_0 than with $D_{0,0}$ or $D_{0,1}$. Also, we can have more stable Type-I ANOVA estimate of σ_2^2 with $D_{0,0}$ than with $D_{0,1}$.
- (b) The variability in the quantile values of $W_{2,1} = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2}$ with respect to (η_1, η_2) is larger for designs $D_{0,0}$ and $D_{0,1}$ than for design D_0 . Moreover, negative estimates are more frequent with the unbalanced designs than with D_0 [see Figure 3.8(a,b)]. Thus a more stable estimate of σ_2^2 is obtained with D_0 than with $D_{0,0}$ or $D_{0,1}$. The QEDs for designs $D_{0,0}$ and $D_{0,1}$ are quite similar although the variability for design $D_{0,0}$ is slightly smaller than that for design $D_{0,1}$ [see Figure 3.8(c)].
- (c) The degree of uniformity in the quantile values of $\hat{W}_{2,1}$ is higher with D_0 than with $D_{0,0}$ or $D_{0,1}$ [see Figure 3.8(a,b)]. However, there is no noticeable difference in the degree of uniformity between $D_{0,0}$ and $D_{0,1}$ [see Figure 3.8(c)].

In summary, the balanced design is strongly preferred to the unbalanced designs for estimating either σ_1^2 or σ_2^2 . The unbalanced design $D_{0,0}$ is slightly preferred over design $D_{0,1}$ for estimating σ_1^2 whereas $D_{0,0}$ is slightly preferred over $D_{0,1}$ for estimating σ_2^2 .

3.3.4 Comparison of designs using ML estimation

In this section, we consider the problem of design comparison for model (3.11) using ML estimation of σ_1^2 or σ_2^2 . Let $\sigma_{1,ML}^2$ and $\sigma_{2,ML}^2$ denote the ML estimation of σ_1^2 or σ_2^2 , respectively. Let us also denote the scaled ML estimation of σ_1^2 and σ_2^2 by $W_{1,ML} = \frac{\sigma_{1,ML}^2}{\sigma_0^2}$ and $W_{2,ML} = \frac{\sigma_{2,ML}^2}{\sigma_0^2}$, respectively. Note that $\sigma_{1,ML}^2$ and $\sigma_{2,ML}^2$ do not have a closed-form expression regardless of whether the data set is balanced or not. Furthermore, the exact distribution of ML estimators is unknown. Therefore, as in Section 2.3.4, we need to use the MCMCs for design comparisons. An outline of the corresponding Monte-Carlo simulation follows:

- (a) Without any loss of generality, we assume that $\mu = 0$ in model (3.11). We then have

$$y_{ijk} = \alpha_i + \beta_j + \gamma_{ijk}, \quad i = 1, 2, \quad j = 1, 2, \quad k = 0, 1, 2, \quad \gamma_{ijk} \sim N(0, \sigma_0^2) \quad (3.12)$$

for the balanced design $\mathcal{D}_{BL}$, $r = s = d$ and $d = 1$. For the unbalanced designs $\mathcal{D}_{UB1}$ and $\mathcal{D}_{UB2}$, $n_{ij} = 3$ or 1 satisfying $N = \sum_{i=1}^2 \sum_{j=1}^2 n_{ij} = 18$. We recall that $\eta_1 = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ and $\eta_2 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$. Hence, $\alpha_i \sim N(0, \sigma_0^2)$, $\beta_j \sim N(0, \eta_1 \sigma_1^2)$ and $\gamma_{ijk} \sim N(0, \sigma_2^2)$. Let us denote a region in the parameter space of the triple $(\alpha_i, \eta_1, \sigma_1^2)$ by V .

- (b) For a given $(\alpha_i, \eta_1, \sigma_1^2) \in V$ and a given design $\mathcal{D}$, a random vector $\mathbf{y}$ is generated on the basis of model (3.11) (i.e. random number) which we denote by y_i is generated from $N(0, \eta_1 \sigma_1^2)$. Independently from y_i , another random number, which we denote by y_j , is generated from $N(0, \eta_2 \sigma_2^2)$. In addition, one random number, which we denote by y_{ijk} is generated from $N(0, \sigma_0^2)$ independently of y_i and y_j . The process produces an observed y_{ijk} value, namely $y_{ijk} = \alpha_i + \beta_j + \gamma_{ijk}$ which represents a response value at the i^{th} row and the j^{th} column. For designs

D_{ML} and D_{ML}^* , the generator of α_{ML} is made only for the $(x, y)^{\text{th}}$ cell of the random matrix $\mathbf{X}$ whose element corresponds to $i \in \{1, \dots, n_{\text{ML}}\}$. The random matrices for $\hat{D}_{\text{ML}}$ and $\hat{D}_{\text{ML}}^*$ were defined in Section 3.3.3. By repeating the process of generating N of α_{ML} 's with appropriate α_1 's and α_2 's, we obtain a vector $\mathbf{g}$ consisting of N observations.

- (i) Using the response vector $\mathbf{g}$ obtained in step (k), ML estimates $\hat{\sigma}_{\text{ML}}^2$ and $\hat{\sigma}_{\text{ML}}^{*2}$ are computed. Values of $\hat{W}_{\text{ML}} = \frac{\hat{\sigma}_{\text{ML}}^2}{\hat{\sigma}_{\text{ML}}^{*2}}$ and $\hat{W}_{\text{ML}}^* = \frac{\hat{\sigma}_{\text{ML}}^2}{\hat{\sigma}_{\text{ML}}^2}$ are then obtained.
- (j) For repeating steps (k) and (i) a sufficient number of times, we can obtain empirical quantiles of $\hat{W}_{\text{ML}}$ and $\hat{W}_{\text{ML}}^*$ for selected values of $\mathbf{p}$ for a given $(\alpha_1, \alpha_2, \sigma_1^2)$. We denote the p^{th} empirical quantile thus obtained by $Q_{\text{ML}}^p(\mathbf{p}, \alpha_1, \alpha_2, \sigma_1^2)$ and $Q_{\text{ML}}^{*p}(\mathbf{p}, \alpha_1, \alpha_2, \sigma_1^2)$ for $\hat{W}_{\text{ML}}$ and $\hat{W}_{\text{ML}}^*$, respectively.
- (k) Steps (i), (j), and (k) are repeated several times corresponding to different values of $(\alpha_1, \alpha_2, \sigma_1^2)$ selected from the region F . The set of such selected triple is denoted by T^* . Thus $T^* \subset F$.
- (l) The maximum and minimum of the p^{th} empirical quantile values over the set T^* in step (k) are obtained for each of $\hat{W}_{\text{ML}}$ and $\hat{W}_{\text{ML}}^*$. We then have, for $\hat{W}_{\text{ML}}$,

$$Q_{\text{ML}}^{\text{max}}(\mathbf{p}) = \max_{(\alpha_1, \alpha_2, \sigma_1^2) \in T^*} Q_{\text{ML}}^p(\mathbf{p}, \alpha_1, \alpha_2, \sigma_1^2), \quad (3.36)$$

$$Q_{\text{ML}}^{\text{min}}(\mathbf{p}) = \min_{(\alpha_1, \alpha_2, \sigma_1^2) \in T^*} Q_{\text{ML}}^p(\mathbf{p}, \alpha_1, \alpha_2, \sigma_1^2), \quad (3.37)$$

and, for $\hat{W}_{\text{ML}}^*$,

$$Q_{\text{ML}}^{*\text{max}}(\mathbf{p}) = \max_{(\alpha_1, \alpha_2, \sigma_1^2) \in T^*} Q_{\text{ML}}^{*p}(\mathbf{p}, \alpha_1, \alpha_2, \sigma_1^2) \quad (3.38)$$

$$Q_{\text{ML}}^{*\text{min}}(\mathbf{p}) = \min_{(\alpha_1, \alpha_2, \sigma_1^2) \in T^*} Q_{\text{ML}}^{*p}(\mathbf{p}, \alpha_1, \alpha_2, \sigma_1^2) \quad (3.39)$$

- (g) Values of $Q_{\text{DGE}}^{(1)}(x)$ and $Q_{\text{DGE}}^{(2)}(x)$ are plotted against x to obtain the EQDCs for W_{DGE} . Also the values of $Q_{\text{DGE}}^{(1)}(x)$ and $Q_{\text{DGE}}^{(2)}(x)$ are plotted against x to obtain the EQDCs for W_{DGE} .

Let us now illustrate the application of the above procedure using the designs, D_{Bd} , D_{DGE} , and D_{DGE} . The region V is chosen as

$$V = \{(y_1, y_2, x_1^2) \mid 0.0 \leq y_1 \leq 10, 0.0 \leq y_2 \leq 10, 0.1 \leq x_1^2 \leq 10\} \quad (2.42)$$

The set T^* considered in step (c) consists of 1120 points selected from V such that

$$T^* = \{(y_1, y_2, x_1^2) \mid y_1 = 0.5(2:1), \text{ and } 1(4:10), y_2 = 0.5(2:1), \text{ and } 1(1:8), \\ x_1^2 = 0.1, 0.5, 1, 5, 10\}$$

A SAS Macro (SAS 1988) was written to compute $Q_{\text{DGE}}^{(1)}(x, y_1, y_2, x_1^2)$ and $Q_{\text{DGE}}^{(2)}(x, y_1, y_2, x_1^2)$ values. The PROC MIXED procedure in SAS (1987) was used to obtain ML estimates of σ_1^2 and σ_2^2 in the SAS Macro program. A total of 1000 replications in step (c) were made to calculate empirical quantile values for each design and each triple (y_1, y_2, x_1^2) in the region T^* . Table 2.8 gives values of $Q_{\text{DGE}}^{(1)}(x)$ and $Q_{\text{DGE}}^{(2)}(x)$ for W_{DGE} . Table 2.9 gives values of $Q_{\text{DGE}}^{(1)}(x)$ and $Q_{\text{DGE}}^{(2)}(x)$ for W_{DGE} . EQDCs plots of W_{DGE} are given in Figure 2.3 and those of W_{DGE} are given in Figure 2.10. We note that the variability in the quantile values for the balanced design D_{Bd} is smaller than those for designs D_{DGE} and D_{DGE} [see Figures 2.3(a,b) and 2.10(a,b)]. However, the distinction between design D_{DGE} and design D_{DGE} is not as much noticeable, even though design D_{DGE} is slightly better than design D_{DGE} in its variability [see Figures 2.9(c) and 2.10(c)]. These observations are consistent with those made with ANOVA estimates in Section 2.2.2. Note that Figures 2.8 and 2.10 appear to be identical. It is because $r = x$ for all designs, $r = x = 0$ for D_{Bd} and $r = x = 10$ for D_{DGE} and D_{DGE} . Also, in the region V in (2.42), ranges of y_1 and y_2 are identical, so that the EQDCs for W_{DGE} must be similar to those for W_{DGE} for all designs.

Table 3-9: Minimum and maximum weighted absolute values of $\mathcal{R}_{\text{dof}}$ for selected values of μ for droplet $\hat{D}_{\text{dof}}$, $\hat{D}_{\text{dof}}$, and $\hat{D}_{\text{dof}}$.

μ	$\hat{D}_{\text{dof}}$		$\hat{D}_{\text{dof}}$		$\hat{D}_{\text{dof}}$	
	min	max	min	max	min	max
0.01	0.000	0.142	-0.000	0.057	0.000	-0.004
0.05	0.000	0.794	-0.000	0.038	0.000	-0.005
0.10	0.001	0.007	-0.000	0.000	0.000	-0.000
0.20	0.001	0.014	0.004	0.710	0.000	-0.000
0.30	0.000	0.007	-0.000	0.010	0.000	-0.000
0.40	0.002	0.000	0.007	0.000	0.000	-0.000
0.50	0.007	0.004	-0.000	1.000	0.000	-0.000
0.60	0.000	1.000	1.000	1.000	1.000	1.000
0.70	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
0.90	1.000	0.004	1.000	0.000	1.000	-0.000
0.95	0.000	0.000	1.000	0.000	1.000	-0.000
0.99	0.000	0.000	1.000	-0.000	1.000	-0.000

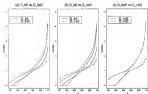


Figure 3.8: SQDCs of $W_{1,0}$ for the three designs, $D_{0.1}$, $D_{0.25}$ and $D_{0.5}$

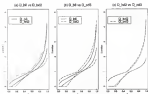


Figure 3.10: QQPlots of $\hat{W}_{D_{22}}$ for the three designs, D_{22} , D_{22} , and D_{22} .

3.2.5. Comparison of ANOVA and ML estimates

We can compare ANOVA estimation with that of ML, with regard to σ_u^2 or σ_v^2 . The comparison, however, is limited since the values $S_{D_{22}, D_{22}}$ in (3.46) and V in (3.43) are not the same. The QQCs for $W_{D_{22}}$ and the QQPlots for $\hat{W}_{D_{22}}$ are shown together in Figure 3.11 for each design. The degree of uniformity in the quantile values of $W_{D_{22}}$ and $\hat{W}_{D_{22}}$ are similar for the balanced design [see Figure 3.11(a)]. However, for the unbalanced designs, D_{22} and D_{22} , the variability in the quantile values of $W_{D_{22}}$ is much smaller than that of $\hat{W}_{D_{22}}$ [see Figure 3.11(b)-(c)]. The QQCs for $W_{D_{22}}$

and the QQQs for W_{ML} are shown in Figure 3.12. It is interesting that the degrees of uniformity in the quantile values for the scaled ANOVA estimator and for the scaled ML estimator are very similar for all designs over the range of ρ . However, the variability in the quantile values with respect to (η_0, η_0) for the scaled ANOVA estimator is smaller than that for the scaled ML estimator, except for the possibility of getting negative values with ANOVA estimation. This holds true for all three designs. In summary, the ML estimator provides much more stable and precise estimate of σ_0^2 than the ANOVA estimator when design is unbalanced. However, in estimating σ_0^2 , the ANOVA estimation produces more precise estimate than that of ML regardless of the unbalancedness of a design.

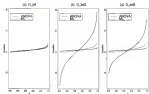


Figure 3.11: QQQs of W_{ML} and PQQQs of W_{ML} for the three designs, D_{ML} , D_{Bal} , and D_{ML} .

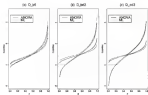


Figure 3.11: QQ-plots of W_{100} and EQQ-plots of W_{100}^* for the three designs, D_{100} , D_{100}^* and D_{100}^{**} .

3.4 The Three-First Nested Random Model

3.4.1 Introduction

A nested design is a design in which the levels of one factor are nested within the levels of another factor. For the balanced nested design, the number of levels of a nested factor within each level of a nesting factor are equal. With the balanced nested design, the allocation of experimental resources yields more information on the lower nested factors than on the higher nested factors. As a result, the design distributes its degrees of freedom unequally so that the higher nested factor has fewer degrees of freedom than the lower nested factor. Therefore, as Lewis et al. (1988, p. 188) indicate, a main disadvantage of balanced nested designs is that the degrees of freedom are too heavily concentrated on the first stage. In order to get more degrees of freedom for the variance associated with the top stage, a considerable increase in the size of the experiment is needed. To alleviate this problem, various modifications were attempted. Balakrishna (1984) introduced partially replicated nested designs, namely the so-called "staggered nested design" and the "a-variant nested design", which keeps the degrees of freedom more evenly in all the nested squares. These designs minimize the variance components more equally efficiently, but result in unbalanced designs. Lewis and Nelson (1988) investigated the empirical distributions of the ANOVA estimates of variance components for the first-stage balanced nested design with several sets of variance components. Lewis et al. (1988) studied the sampling distributions of the ANOVA estimates of variance components for the first-stage staggered nested design as well as the first-stage factorial nested design as an extension of the work by Lewis and Nelson (1988). They concluded that the design used had little effect on the shape of the resulting distributions of the variance components estimators, and that no design was best for estimating all the variance components. However, the staggered nested design was recommended as a good compromise choice because this

design spreads the degrees of freedom almost evenly over all the stages. They also mentioned that this design is not only simple to administer but is also "representative", which means "the design allows some levels of the top stage variable to be added without upsetting the analysis" (Leone et al., 1988, p. 122). It is, however, still of interest to know how good the staggered nested design is as compared to the balanced nested design, or the limited nested design.

The estimation problem of the variance components for the staggered nested design has been addressed by Fienberg (1981, 1982), Smith and Harvey (1984), Nelson (1982, 1986a, 1986b), and more recently by Klarion, Niek, and Mease (1987). Nelson (1986a) asserts that the staggered nested design is the most popular among all nested designs. Whitmore and Niek (1986) presented statistical tests for variance components in the staggered nested design.

Figure 1 illustrates part of the first stage balanced nested design, the four-stage staggered nested design, and the four-stage nested nested design. In this section, these three designs will be compared on a non-comparative way, namely by looking at the dispersion of the quantities of the variance components estimators. The designs compared have the same number of observations. The degrees of freedom for the designs will be shown later in their ANOVA tables.

3.4.1. Notations and model

Consider the unbalanced three-fold nested random model,

$$y_{ijkl} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} + \epsilon_{ijkl} \quad (3.41)$$

$i=1,2, \dots, n, j=1,2, \dots, J_i, k=1,2, \dots, K_{ij}, l=1,2, \dots, K_{ijk}$, where $\sum_{i=1}^n K_i = n$, $\sum_{i=1}^n \sum_{j=1}^{J_i} K_{ij} = n$, and $\sum_{i=1}^n \sum_{j=1}^{J_i} \sum_{k=1}^{K_{ij}} K_{ijk} = N$. Let α denote the factors of first three stages as $A, B(A),$ and $C(AB)$, respectively. In model (3.41), μ is an overall mean, α_i is the effect due to the i^{th} level of factor A , β_{ij} is the effect due to the j^{th}



Figure 4.13: Four-Stage Nested Design: (a) the Balanced Nested Design Showing The First and Last Levels of the Stage 1, (b) the Saturated Nested Design Showing The First and Last Levels of the Stage 1, and (c) the Unbalanced Nested Design Showing The First Four Levels of the Stage 1. [Adapted from Lerner et al. (1983)]

level of factor B nested within the i^{th} level of factor A, γ_{ijk} is the effect due to the j^{th} level of factor C nested within the j^{th} level of factor B which is also nested within the i^{th} level of factor A, and ϵ_{ijk} is a random experimental error. We assume that μ is a fixed unknown parameter and α_i 's, β_{ijk} 's, γ_{ijk} 's, and ϵ_{ijk} 's are independently distributed such that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\beta_{ijk} \sim N(0, \sigma_{\beta_{ijk}}^2)$, $\gamma_{ijk} \sim N(0, \sigma_{\gamma_{ijk}}^2)$, and $\epsilon_{ijk} \sim N(0, \sigma_\epsilon^2)$. The model is said to be balanced if the range of any subscript of a nested factor does not depend on the values of a nesting factor. Model (3-12) can be written in vector form as

$$\mathbf{y} = \mu \mathbf{1}_N + \mathbf{X}_1 \alpha + \mathbf{X}_2 \beta + \mathbf{X}_3 \gamma + \mathbf{z}, \quad (3-14)$$

where

$\mathbf{1}_N$ = the vector of ones of order $N \times 1$,

$$\mathbf{X}_1 = \mathbf{B}_{ijk}^T \mathbf{1}_{N_1}, \text{ where } N_1 = \sum_{j=1}^{J_1} \sum_{k=1}^{J_2} n_{ijk},$$

$$\mathbf{X}_2 = \mathbf{B}_{ijk}^T \mathbf{B}_{jkl}^T \mathbf{1}_{N_2}, \text{ where } N_2 = \sum_{k=1}^{J_2} n_{ijk},$$

$$\mathbf{X}_3 = \mathbf{B}_{ijk}^T \times \mathbf{B}_{jkl}^T \mathbf{B}_{lmn}^T \mathbf{1}_{N_3},$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{J_1})^T,$$

$$\beta = (\beta_{111}, \beta_{112}, \dots, \beta_{J_1 J_2 J_3})^T,$$

$$\gamma = (\gamma_{111}, \gamma_{112}, \dots, \gamma_{J_1 J_2 J_3})^T,$$

and $\mathbf{z}$ is defined analogously to $\mathbf{y}$ as $(N \times 1)$ vector of observations. The mean of $\mathbf{y}$ is $\mu \mathbf{1}_N$ and the variance-covariance matrix, denoted by Σ , is of the form

$$\Sigma = \mathbf{X}_1 \mathbf{X}_1^T \sigma_\alpha^2 + \mathbf{X}_2 \mathbf{X}_2^T \sigma_{\beta_{ijk}}^2 + \mathbf{X}_3 \mathbf{X}_3^T \sigma_{\gamma_{ijk}}^2 + \mathbf{I}_N \sigma_\epsilon^2 \quad (3-15)$$

In this study, we consider the quantiles of the distributions of $\sum_{i=1}^N y_i$, $\sum_{i=1}^N y_i^2$, and $\sum_{i=1}^N y_i^3$ for comparing designs as well as comparing two estimators not before, namely, ANOVA

and ML, where σ_{α}^2 , $\sigma_{\beta_{\text{ind}}}^2$, and $\sigma_{\gamma_{\text{ind}}}^2$ denote an estimator of σ_{α}^2 , $\sigma_{\beta_{\text{ind}}}^2$, and $\sigma_{\gamma_{\text{ind}}}^2$ respectively. Three designs, namely a four-stage balanced nested design, a four-stage staggered nested design, and a four-stage unreplicated nested design are considered. The total number of observations for each design is 48. These designs are the ones used by Loefer and Nelson (1993) and Loefer et al. (1995), so that we can compare our results with theirs. ANOVA estimators are considered in Sections 3.4.3 and 3.4.4 whereas Section 3.4.5 is focused on ML estimators. Comparisons of these estimation methods for each design are made in Section 3.4.6.

3.4.3. Statistics of $\hat{\sigma}_{\alpha}^2$, $\hat{\sigma}_{\beta_{\text{ind}}}^2$, and $\hat{\sigma}_{\gamma_{\text{ind}}}^2$ using ANOVA estimators

An ANOVA table for model (3.40) is of the form

Source	d.f.	Sum of Squares	Expected Mean Squares
A	$a - 1$	SS_A	$k_1\sigma_{\alpha}^2 + k_2\sigma_{\beta_{\text{ind}}}^2 + k_3\sigma_{\gamma_{\text{ind}}}^2 + \sigma_{\epsilon}^2$
B(A)	$b - a$	$SS_{B(A)}$	$k_2\sigma_{\beta_{\text{ind}}}^2 + k_3\sigma_{\gamma_{\text{ind}}}^2 + \sigma_{\epsilon}^2$
C(AB)	$c - b$	$SS_{C(AB)}$	$k_3\sigma_{\gamma_{\text{ind}}}^2 + \sigma_{\epsilon}^2$
Error	$N - c$	SS_e	σ_{ϵ}^2
Total(48)	$N - 1$	SS_T	

where

$$SS_A = \mathbf{y}'\mathbf{Q}_A\mathbf{y}$$

$$SS_{B(A)} = \mathbf{y}'\mathbf{Q}_{B(A)}\mathbf{y}$$

$$SS_{C(AB)} = \mathbf{y}'\mathbf{Q}_{C(AB)}\mathbf{y}$$

$$SS_e = \mathbf{y}'\mathbf{Q}_e\mathbf{y}$$

with

$$Q_1 = \sigma_{10}^2 \left(\frac{1}{n_1} J_{n_1} \right) = \frac{1}{J} J_2, \quad (3.46)$$

$$Q_2 = \sigma_{10}^2 \sigma_{11}^2 \left(\frac{1}{n_1} J_{n_1} \right) = \sigma_{10}^2 \left(\frac{1}{J} J_{n_1} \right), \quad (3.47)$$

$$Q_3 = \sigma_{10}^2 \sigma_{11}^2 \sigma_{12}^2 \left(\frac{1}{n_{12}} J_{n_{12}} \right) = \sigma_{10}^2 \sigma_{11}^2 \left(\frac{1}{n_1} J_{n_{12}} \right), \text{ and } (3.48)$$

$$R = J_2 = \sigma_{10}^2 \sigma_{11}^2 \sigma_{12}^2 \left(\frac{1}{n_{12}} J_{n_{12}} \right). \quad (3.49)$$

The coefficients of the variance components in the expressions for the expected mean squares are (Makridakis, 1983):

$$b_1 = \frac{1}{s-1} \sum_{i=1}^s n_i^2 \left(\frac{1}{n_i} - \frac{1}{J} \right), \quad (3.50)$$

$$b_2 = \frac{1}{s-1} \sum_{i=1}^s \sum_{j=1}^k n_{ij}^2 \left(\frac{1}{n_i} - \frac{1}{J} \right), \quad (3.51)$$

$$b_3 = \frac{1}{s-1} \sum_{i=1}^s \sum_{j=1}^k \sum_{l=1}^{n_{ij}} n_{ijl}^2 \left(\frac{1}{n_i} - \frac{1}{J} \right), \quad (3.52)$$

$$b_4 = \frac{1}{s-1} \sum_{i=1}^s \sum_{j=1}^k n_{ij}^2 \left(\frac{1}{n_{ij}} - \frac{1}{n_i} \right), \quad (3.53)$$

$$b_5 = \frac{1}{s-1} \sum_{i=1}^s \sum_{j=1}^k \sum_{l=1}^{n_{ij}} n_{ijl}^2 \left(\frac{1}{n_{ij}} - \frac{1}{n_i} \right) \text{ and } (3.54)$$

$$b_6 = \frac{1}{s-1} \sum_{i=1}^s \sum_{j=1}^k \sum_{l=1}^{n_{ij}} n_{ijl}^2 \left(\frac{1}{n_{ijl}} - \frac{1}{n_{ij}} \right). \quad (3.55)$$

The mean squares of the factors are denoted by MS_{α} , $MS_{\alpha\beta}$, $MS_{\alpha\beta\gamma}$, and MS_{α} accordingly. Let us denote ANOVA estimators of σ_{10}^2 , $\sigma_{10\beta}^2$, $\sigma_{10\beta\gamma}^2$ by $\hat{\sigma}_{10}^2$, $\hat{\sigma}_{10\beta}^2$ and $\hat{\sigma}_{10\beta\gamma}^2$, respectively. Let us also denote the scaled variances of these estimators

by $W_{\text{tot}} = \frac{d^2}{dt^2}$, $W_{\text{tot}(\text{tot})} = \frac{d^2}{dt^2}W_{\text{tot}}$, and $W_{\text{tot}(\text{tot}(\text{tot}))} = \frac{d^2}{dt^2}W_{\text{tot}(\text{tot})}$, respectively. Using Theorem 3.3.1 and Corollary 3.3.1, we can obtain the distribution of $\hat{W}_{\text{tot}}$, $\hat{W}_{\text{tot}(\text{tot})}$ and $\hat{W}_{\text{tot}(\text{tot}(\text{tot}))}$ which are shown in the following theorem.

Theorem 3.3.1. For the unbalanced three-fold nested random model (3.15), the scaled ANOVA estimators of σ_0^2 , $\sigma_{\text{tot}(\text{tot})}^2$ and $\sigma_{\text{tot}(\text{tot}(\text{tot}))}^2$ are distributed as

$$\hat{W}_{\text{tot}} \stackrel{d}{=} \sum_{j=1}^J \gamma_j \hat{R}_j^2, \quad (3.76)$$

$$\hat{W}_{\text{tot}(\text{tot})} \stackrel{d}{=} \sum_{j=1}^J \hat{R}_j \hat{R}_{\text{tot}(\text{tot})}^2, \quad (3.77)$$

$$\hat{W}_{\text{tot}(\text{tot}(\text{tot}))} \stackrel{d}{=} \sum_{j=1}^J \hat{R}_j \hat{R}_{\text{tot}}^2, \quad (3.78)$$

where

J = number of distinct positive eigenvalues of $\mathbf{Q}_0 \mathbf{X}_0$

J' = number of distinct positive eigenvalues of $\mathbf{Q}_0 \mathbf{X}_0$

J'' = number of distinct positive eigenvalues of $\mathbf{Q}_0 \mathbf{X}_0$

γ_j = j^{th} positive eigenvalue of $\mathbf{Q}_0 \mathbf{X}_0$ with multiplicity γ_j , $j = 1, 2, \dots, J'$

$\hat{R}_j$ = j^{th} positive eigenvalue of $\mathbf{Q}_0 \mathbf{X}_0$ with multiplicity m_j , $j = 1, 2, \dots, J''$

$\hat{R}_j$ = j^{th} positive eigenvalue of $\mathbf{Q}_0 \mathbf{X}_0$ with multiplicity r_j , $j = 1, 2, \dots, J''$

$$\hat{R}_j = \frac{\sigma_0^2}{\sigma_j^2},$$

$$\hat{R}_j = \frac{\sigma_0^2 \sigma_j^2}{\sigma_j^2},$$

$$\hat{R}_j = \frac{\sigma_0^2 \sigma_j^2}{\sigma_j^2}.$$

$$\begin{aligned} \hat{Q}_1 = & \frac{1}{k_1(N-1)}\hat{Q}_1 - \frac{k_1}{k_1k_2(N-1)}\hat{Q}_2 - \frac{(k_1k_2-k_2k_1)}{k_1k_2k_3(N-1)}\hat{Q}_3 \\ & - \frac{(k_1k_2+k_1k_3-k_2k_3-k_2k_1)}{k_1k_2k_3(N-1)}\mathbf{R}, \end{aligned} \quad (3.35)$$

$$\hat{Q}_2 = \frac{1}{k_2(N-1)}\hat{Q}_2 - \frac{k_2}{k_1k_2(N-1)}\hat{Q}_1 - \frac{(k_2-k_1)}{k_2k_3(N-1)}\mathbf{R}, \quad (3.36)$$

$$\hat{Q}_3 = \frac{1}{k_3(N-1)}\hat{Q}_3 - \frac{1}{k_3(N-1)}\mathbf{R}. \quad (3.37)$$

$$\mathbf{E}_1 = \frac{1}{x_1^2}\mathbf{E} = \mathbf{E}_1\mathbf{E}_1 + \mathbf{E}_1\mathbf{E}_2\left(\frac{1}{x_1}\right) + \mathbf{E}_1\mathbf{E}_3\left(\frac{1}{x_1}\right) + \mathbf{E}_2\left(\frac{1}{x_1}\right), \quad (3.38)$$

$$\mathbf{E}_2 = \frac{1}{x_2^2}\mathbf{E} = \mathbf{E}_2\mathbf{E}_1\left(\frac{1}{x_2}\right) + \mathbf{E}_2\mathbf{E}_2 + \mathbf{E}_2\mathbf{E}_3\left(\frac{1}{x_2}\right) + \mathbf{E}_3\left(\frac{1}{x_2}\right), \quad (3.39)$$

$$\mathbf{E}_3 = \frac{1}{x_3^2}\mathbf{E} = \mathbf{E}_3\mathbf{E}_1\left(\frac{1}{x_3}\right) + \mathbf{E}_3\mathbf{E}_2\left(\frac{1}{x_3}\right) + \mathbf{E}_3\mathbf{E}_3 + \mathbf{E}_3\left(\frac{1}{x_3}\right). \quad (3.40)$$

Here, the matrices $\hat{Q}_1$, $\hat{Q}_2$, $\hat{Q}_3$, and $\mathbf{R}$ are defined in (3.35) through (3.37), and k_i 's, $i = 1, 2, \dots, 3$, are the coefficients of the variance components in the repeated mean system, which are defined in (3.38) through (3.39).

Proof. From the repeated mean system given in the ANOVA table, the ANOVA estimators of $\sigma_1^2 + \sigma_2^2 = MS_B = \frac{1}{2}\left(\frac{\hat{\mathbf{R}}}{x_1^2}\right)'$. The ANOVA estimator of σ_{total}^2 , σ_{total}^2 and σ_3^2 are given by

$$\begin{aligned} \sigma_{\text{total}}^2 &= \frac{1}{k_1}(MS_{\text{Total}} - MS_B) \\ &= \frac{1}{2}\left[\frac{1}{k_1(N-1)}\hat{Q}_1 - \frac{1}{k_1(N-1)}\mathbf{R}\right]' \\ &= \frac{1}{2}\hat{Q}_1\mathbf{R}. \end{aligned}$$

$$\begin{aligned}
v_{\text{low}}^2 &= \frac{1}{k_1} (W_{\text{low}}^2 - \frac{k_2}{k_1} W_{\text{low}} W_{\text{high}} - W_{\text{high}}^2) - M_{\text{low}}^2 \\
&= \mathbf{y} \left[\frac{1}{k_1(N-a)} \mathbf{Q}_0 - \frac{k_2}{k_1 k_2(N-1)} \mathbf{Q}_1 - \frac{(k_1-k_2)}{k_1 k_2(N-1)} \mathbf{R} \right] \mathbf{y} \\
&= \mathbf{y}^T \mathbf{Q}_0 \mathbf{y} \quad \text{and} \\
v_{\text{high}}^2 &= \frac{2}{k_1} (W_{\text{high}}^2 - \frac{k_2}{k_1} M_{\text{low}} W_{\text{high}} + \frac{k_2}{k_1} M_{\text{high}} W_{\text{low}} - M_{\text{low}}^2) \\
&\quad - \frac{k_2}{k_1} (M_{\text{high}} W_{\text{high}} - M_{\text{high}}^2) - M_{\text{high}}^2 \\
&= \mathbf{y} \left[\frac{1}{k_1(N-1)} \mathbf{Q}_0 - \frac{k_2}{k_1 k_2(N-a)} \mathbf{Q}_1 - \frac{(k_2 k_1 - k_1 k_2)}{k_1 k_2 k_2(N-1)} \mathbf{Q}_2 \right. \\
&\quad \left. - \frac{(k_2 k_1 + k_2 k_2 - k_1 k_2 - k_1 k_2)}{k_1 k_2 k_2(N-a)} \mathbf{R} \right] \mathbf{y} \\
&= \mathbf{y}^T \mathbf{Q}_1 \mathbf{y},
\end{aligned}$$

respectively. The matrices $\mathbf{Q}_0$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ are symmetric positive semidefinite. Moreover, $\mathbf{Q}_0 \mathbf{1}_N = \mathbf{Q}_1 \mathbf{1}_N = \mathbf{Q}_2 \mathbf{1}_N = \mathbf{0}$. By letting $\mathbf{E}_0 = \frac{1}{N} \mathbf{I}$, $\mathbf{E}_1 = \frac{1}{N(N-1)} \mathbf{Z}$, and $\mathbf{E}_2 = \frac{1}{N(N-1)} \mathbf{Z}$ where $\mathbf{Z}$ is defined in (16), and by applying Corollary 3.2.1, the scaled ANOVA estimators of v_{low}^2 , v_{high}^2 , and $v_{\text{low},\text{high}}^2$ are distributed as the theorem states. This completes the proof of the theorem. $\square$

Corollary 3.4.1. Consider the balanced three-fold period random model,

$$Y_{ijkl} = \mu + \alpha_i + \beta_{jkl} + \gamma_{ijk} + \epsilon_{ijkl}, \quad (3.15)$$

$i = 1, 2, \dots, m$, $j = 1, 2, \dots, k$, $k = 1, 2, \dots, N$, and $l = 1, 2, \dots, N$, where $N = mn$.

Then, $W_{\text{low}} = \frac{1}{N} \mathbf{y}^T \mathbf{W}_{\text{low}} \mathbf{y}$, $W_{\text{high}} = \frac{1}{N} \mathbf{y}^T \mathbf{W}_{\text{high}} \mathbf{y}$, and $W_{\text{low},\text{high}} = \frac{1}{N} \mathbf{y}^T \mathbf{W}_{\text{low},\text{high}} \mathbf{y}$ are distributed as

$$\begin{aligned} W_{\text{out}} &\stackrel{\text{d}}{=} \frac{1}{(x-1)\log} \left[\log + n \left(\frac{2n}{n} \right) + n \left(\frac{2n}{n} \right) + \left(\frac{1}{n} \right) \right] x_{\text{out}}^2 \\ &= \frac{1}{2(x-1)\log} \left[n \left(\frac{2n}{n} \right) + n \left(\frac{2n}{n} \right) + \left(\frac{1}{n} \right) \right] x_{\text{out}+n}^2. \end{aligned} \quad (4.88)$$

$$\begin{aligned} W_{\text{out}+n} &\stackrel{\text{d}}{=} \frac{1}{2(x-1)\log} \left[n + n \left(\frac{n}{n} \right) + \left(\frac{1}{n} \right) \right] x_{\text{out}+n}^2 \\ &= \frac{1}{2(x-1)\log} \left[n \left(\frac{2n}{n} \right) + \left(\frac{1}{n} \right) \right] x_{\text{out}+n}^2 \text{ and} \end{aligned} \quad (4.89)$$

$$\begin{aligned} W_{\text{out}+n} &\stackrel{\text{d}}{=} \frac{1}{2(x-1)\log} \left(n + \frac{1}{n} \right) x_{\text{out}+n}^2 \\ &= \frac{1}{2n(x-1)} \left(\frac{1}{n} \right) x_{\text{out}+n}^2, \end{aligned} \quad (4.90)$$

respectively, where $\eta_n = \frac{1}{n}$, $\eta_n = \frac{1}{2n}$ and $\eta_n = \frac{1}{2n}$.

Proof. With model (2.10), the expected cross squares of the factors are

$$E(WX_k) = \log x_k^2 + n\sigma_{\log}^2 + n\sigma_{\log}^2 + \sigma_k^2, \quad E(WX_{n,n}) = n\sigma_{\log}^2 + n\sigma_{\log}^2 + \sigma_k^2,$$

$$E(WX_{n,n}) = n\sigma_{\log}^2 + \sigma_k^2, \text{ and } E(WX_k) = \sigma_k^2. \text{ Hence, the sums of squares of}$$

$$\text{the factors are distributed as } WX_k \sim E(MX_k)x_{\text{out}+n}^2, \quad WX_{n,n} \sim E(MX_{n,n})x_{\text{out}+n}^2,$$

$$WX_{1,n} \sim E(MX_{1,n})x_{\text{out}+n}^2, \text{ and } WX_k \sim E(WX_k)x_{\text{out}+n}^2. \text{ Moreover, they are}$$

independently distributed. Therefore, the CNRM estimates of unknown components

are

$$\begin{aligned} x_{\text{in}}^2 &= \frac{1}{\log}(WX_k - E(WX_k)) \\ &\stackrel{\text{d}}{=} \frac{1}{\log} \left[\frac{E(WX_k)}{x-1} x_{\text{in}}^2 + \frac{E(MX_k)}{2(x-1)} x_{\text{out}+n}^2 \right], \end{aligned}$$

$$\begin{aligned}
\hat{r}_{\text{MD},k}^{\text{MD}} &= \frac{1}{m} (M\hat{F}_{\text{MD},k} - M\hat{F}_{\text{MD},k-1}) \\
&\leq \frac{1}{m} \left[\frac{E(M\hat{F}_{\text{MD},k})}{m(k-1)} \hat{r}_{\text{MD},k-1}^{\text{MD}} - \frac{E(M\hat{F}_{\text{MD},k})}{m(k-1)} \hat{r}_{\text{MD},k-1}^{\text{MD}} \right], \text{ and} \\
\hat{r}_{\text{MD},k,k}^{\text{MD}} &= \frac{1}{n} (M\hat{F}_{\text{MD},k,k} - M\hat{F}_k) \\
&\leq \frac{1}{n} \left[\frac{E(M\hat{F}_{\text{MD},k,k})}{m(k-1)} \hat{r}_{\text{MD},k-1}^{\text{MD}} - \frac{E(M\hat{F}_k)}{m(k-1)} \hat{r}_{\text{MD},k-1}^{\text{MD}} \right]
\end{aligned}$$

Since $\hat{F}_{k,k} = \frac{1}{m} W_{k,k,k}$, $\hat{F}_{k,k,k} = \frac{1}{m} w_{k,k,k}$, and $\hat{F}_{\text{MD},k,k} = \frac{1}{m} w_{\text{MD},k,k}$, the result follows. $\square$

The exact distributions of $\hat{F}_{k,k}$, $\hat{F}_{\text{MD},k,k}$, and $\hat{F}_{\text{MD},k,k}$ in Theorem 3.4.1, and in Corollary 3.4.1, are determined by their quantiles, whose exact values are obtained by using, for example, Davies' (1980) algorithm.

3.4.1 Computation of design using ADOA estimation

The quantiles of $\hat{F}_{k,k}$, $\hat{F}_{\text{MD},k,k}$, and $\hat{F}_{\text{MD},k,k}$ depend on the unknown values of $\eta_1 = \frac{\sigma_1^2}{\sigma_2^2}$, $\eta_2 = \frac{\sigma_2^2}{\sigma_3^2}$, and $\eta_3 = \frac{\sigma_3^2}{\sigma_4^2}$ as well as on the design employed. We shall denote a design by D . Let us denote $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$. The dependency of the p^{th} quantile on these unknown values and the design is indicated by writing $q(p, \boldsymbol{\eta})$ for $\hat{F}_{k,k}$, $q_1(p, \boldsymbol{\eta})$ for $\hat{F}_{\text{MD},k,k}$, and $q_2(p, \boldsymbol{\eta})$ for $\hat{F}_{\text{MD},k,k}$. Let us also denote a region in the parameter space of $\boldsymbol{\eta}$ by $S_{\boldsymbol{\eta}}$. The QOCs of $\hat{F}_{k,k}$, $\hat{F}_{\text{MD},k,k}$, and $\hat{F}_{\text{MD},k,k}$ are obtained by plotting their maximum and minimum quantiles against p , which are defined as follows.

For $\hat{F}_{k,k}$,

$$Q_{\text{OC}}^{\text{MD}}(p) = \max_{\boldsymbol{\eta} \in S_{\boldsymbol{\eta}}} q(p, \boldsymbol{\eta}), \text{ and} \quad (3.46)$$

$$Q_{\text{OC}}^{\text{MD}}(p) = \min_{\boldsymbol{\eta} \in S_{\boldsymbol{\eta}}} q(p, \boldsymbol{\eta}). \quad (3.47)$$

For $\bar{W}_{\text{factor}}$

$$Q_{\text{QO}}^{\text{QO}}(y) = \frac{1}{q_1^2 q_2^2} \phi_1^2(x, y), \text{ and} \quad (3.18)$$

$$Q_{\text{QO}}^{\text{QO}}(y) = \frac{1}{q_1^2 q_2^2} \phi_2^2(x, y). \quad (3.19)$$

For $\bar{W}_{\text{factor}}$

$$Q_{\text{QO}}^{\text{QO}}(y) = \frac{1}{q_1^2 q_2^2} \phi_1^2(x, y), \text{ and} \quad (3.20)$$

$$Q_{\text{QO}}^{\text{QO}}(y) = \frac{1}{q_1^2 q_2^2} \phi_2^2(x, y). \quad (3.21)$$

As it is stated in Section 3.1.1, three designs, namely a four-stage balanced nested design, a four-stage staggered nested design, and a four-stage crossed nested design are to be compared. We denote these designs by D_{bal} , D_{stg} , and D_{crs} , respectively. The estimation capability of these designs, using ANOVA estimation, with regard to σ_1^2 , σ_{bal}^2 , and σ_{stg}^2 will be compared using the QOCs.

The balanced nested design, D_{bal}

The ANOVA table for model (3.6) using design D_{bal} with $a = 3$, $b = 2$, $c = 2$, and $n = 3$ is

Source	d.f	Base of Squares	Expected Mean Squares
A	4	$\frac{1}{b} Q_{A,12}$	$3\sigma_1^2 + 4\sigma_{\text{bal}}^2 + 3\sigma_{\text{crs}}^2 + \sigma_\epsilon^2$
B(A)	8	$\frac{1}{b} Q_{B,12}$	$4\sigma_{\text{bal}}^2 + 3\sigma_{\text{crs}}^2 + \sigma_\epsilon^2$
C(AB)	16	$\frac{1}{b} Q_{C,12}$	$3\sigma_{\text{crs}}^2 + \sigma_\epsilon^2$
Error	36	$\frac{1}{b} R_{12}$	σ_ϵ^2
Total(only)	56	RT_2	

From Corollary 3.4.1, the scaled ANOVA estimators for σ_{μ}^2 , $\sigma_{\beta_{\text{row}}}^2$ and $\sigma_{\beta_{\text{col}}}^2$ are described as

$$W_{\text{row}} \stackrel{\text{d}}{=} \frac{1}{20} \left[3 + 3 \left(\frac{20}{b_1} \right) + 3 \left(\frac{20}{b_2} \right) + \frac{1}{b_3} \right] x_1^2 \quad (2.67)$$

$$+ \frac{1}{20} \left[3 \left(\frac{20}{b_1} \right) + 3 \left(\frac{20}{b_2} \right) + \frac{1}{b_3} \right] x_2^2 \quad (2.68)$$

$$W_{\text{total}} \stackrel{\text{d}}{=} \frac{1}{20} \left[4 + 3 \left(\frac{20}{b_1} \right) + \frac{1}{b_3} \right] x_1^2 + \frac{1}{20} \left[3 \left(\frac{20}{b_1} \right) + \frac{1}{b_3} \right] x_2^2 \text{ and } (2.69)$$

$$W_{\text{column}} \stackrel{\text{d}}{=} \frac{1}{20} \left[3 + \frac{1}{b_3} \right] x_3^2 + \frac{1}{20 b_3} x_4^2 \text{ respectively} \quad (2.68)$$

The diagonal mixed design, D_{20}

The ANOVA table for model (2.63) using design D_{20} with $a = 10$, $b = 10$, $c = 10$ and $R = 10$ is

Source	d.f.	Sum of Squares	Expected Mean Squares
μ	1	$\mu' Q_{\mu} \mu$	$4\sigma_{\mu}^2 + \frac{1}{3}\sigma_{\beta_{\text{row}}}^2 + \frac{1}{3}\sigma_{\beta_{\text{col}}}^2 + \sigma_{\epsilon}^2$
β_{row}	10	$\mu' Q_{\beta_{\text{row}}} \mu$	$\frac{1}{3}\sigma_{\beta_{\text{row}}}^2 + \frac{1}{3}\sigma_{\beta_{\text{col}}}^2 + \sigma_{\epsilon}^2$
β_{col}	10	$\mu' Q_{\beta_{\text{col}}} \mu$	$\frac{1}{3}\sigma_{\beta_{\text{row}}}^2 + \sigma_{\epsilon}^2$
Total	10	$\mu' B \mu$	σ_{ϵ}^2
Total(adj.)	10	$B \beta$	

Model (2.63) as vector form for design D_{20} can be written as

$$y = \mu A_0 + (F_{\text{row}} \otimes I_2) \alpha + (F_{\text{col}} \otimes I_2) \beta + (F_{\text{row}} \otimes B) \gamma + \epsilon,$$

where $A_0 = I_2 \otimes I_2$ and $B = I_2 \otimes I_2$. The variance-covariance matrix of y , E is

$$E = (F_{\text{row}} \otimes I_2) \sigma_{\alpha}^2 + (F_{\text{col}} \otimes A_0) \sigma_{\beta_{\text{row}}}^2 + (F_{\text{row}} \otimes B B) \sigma_{\beta_{\text{col}}}^2 + F_{\text{row}} \sigma_{\epsilon}^2.$$

and, hence, Σ_0 , Σ_1 , and Σ_2 in (3.65) through (3.67) are

$$\Sigma_0 = (I_D \oplus I_1) + (J_{12} \oplus A, A') \begin{pmatrix} \frac{20}{120} \end{pmatrix} + (J_{12} \oplus B, B') \begin{pmatrix} \frac{20}{120} \end{pmatrix} + J_{22} \begin{pmatrix} \frac{1}{120} \end{pmatrix},$$

$$\Sigma_1 = (I_D \oplus I_1) \begin{pmatrix} \frac{20}{120} \end{pmatrix} + (J_{12} \oplus A, A') + (J_{12} \oplus B, B') \begin{pmatrix} \frac{20}{120} \end{pmatrix} + J_{22} \begin{pmatrix} \frac{1}{120} \end{pmatrix}, \text{ and}$$

$$\Sigma_2 = (I_D \oplus I_1) \begin{pmatrix} \frac{20}{120} \end{pmatrix} + (J_{12} \oplus A, A') \begin{pmatrix} \frac{20}{120} \end{pmatrix} + (J_{12} \oplus B, B') + J_{22} \begin{pmatrix} \frac{1}{120} \end{pmatrix},$$

respectively. The matrices of $\hat{Q}_0$, $\hat{Q}_1$, and $\hat{Q}_2$ in equations (3.71) through (3.81), for this design, are

$$\hat{Q}_0 = \frac{1}{120}\hat{Q}_0 + \frac{1}{120}\hat{Q}_0 + \frac{1}{120}\hat{Q}_0 + \frac{1}{120}\hat{R},$$

$$\hat{Q}_1 = \frac{1}{120}\hat{Q}_0 + \frac{7}{120}\hat{Q}_0 + \frac{1}{120}\hat{R},$$

$$\hat{Q}_2 = \frac{2}{20}\hat{Q}_0 + \frac{2}{20}\hat{R},$$

where

$$\hat{Q}_0 = \frac{1}{4}(J_D \oplus J_D) + \frac{1}{40}J_{22}$$

$$\hat{Q}_1 = I_{22} \oplus A' + \frac{1}{4}(J_{12} \oplus J_{12}),$$

$$\hat{Q}_2 = I_{22} \oplus B' + I_D \oplus A'$$

$$\hat{R} = I_{22} - I_D \oplus B',$$

and $A' = \frac{1}{2}J_1 \oplus I_1$ and $B' = \frac{1}{2}J_1 \oplus I_1$. The matrices, $\hat{Q}_0$, $\hat{Q}_1$, $\hat{Q}_2$, and $\hat{R}$, are obtained by using equations (3.68) through (3.70) with design D_{opt} . By applying Theorem 3.4.1 with the matrices, $\hat{Q}_0$, $\hat{Q}_1$, $\hat{Q}_2$, Σ_0 , Σ_1 , and Σ_2 , the distribution of $\hat{W}_{0,1}$, $\hat{W}_{1,0,1}$, and $\hat{W}_{2,0,1}$ for the design D_{opt} are obtained.

The localized nested design, $D_{N\alpha}$

Similarly, for design $D_{N\alpha}$, with $n = 10$, $k = 20$, $s = 20$, and $N = 40$, the $AN(P, \lambda)$ table is

Source	df	Sum of Squares	Expected Mean Square
A	10	$nQ_{A,0}$	$\frac{1}{2}r_1^2 + \frac{10}{20}r_{A,0}^2 + \frac{10}{20}r_{A,20}^2 + r_1^2$
B(A)	10	$nQ_{B,0}$	$\frac{1}{2}r_{A,0}^2 + \frac{1}{2}r_{A,20}^2 + r_1^2$
C(AB)	1	$nQ_{C,0}$	$\frac{1}{2}r_{A,0}^2 + r_1^2$
Error	4	nW_0	r_1^2
Total(adj.)	30	kD_7	

The model is

$$y = \mu A_0 + (I_1 \oplus C^*)m + (I_1 \oplus D)pl + (I_1 \oplus E)\gamma + \alpha,$$

where $C^* = I_2 \oplus I_2 \oplus I_2 \oplus I_2$, $D = I_2 \oplus I_2 \oplus I_2 \oplus I_2$, and $E = I_2 \oplus I_2$. The variance-covariance matrix of y , Σ , is

$$\Sigma = (I_1 \oplus C^*C^*)r_1^2 + (I_1 \oplus D.D)r_{A,0}^2 + (I_2 \oplus E.E)r_{A,20}^2 + I_{40}r_1^2$$

and hence E_1 , E_2 , and E_3 in (3.32) through (3.34) are

$$E_1 = (I_1 \oplus C^*C^*) + (I_1 \oplus D.D) \left(\frac{n}{n_0} \right) + (I_1 \oplus E.E) \left(\frac{n}{n_0} \right) + I_{40} \left(\frac{1}{n_0} \right),$$

$$E_2 = (I_1 \oplus C^*C^*) \left(\frac{n}{n_0} \right) + (I_1 \oplus D.D) + (I_1 \oplus E.E) \left(\frac{n}{n_0} \right) + I_{40} \left(\frac{1}{n_0} \right), \text{ and}$$

$$E_3 = (I_1 \oplus C^*C^*) \left(\frac{n}{n_0} \right) + (I_1 \oplus D.D) \left(\frac{n}{n_0} \right) + (I_2 \oplus E.E) + I_{40} \left(\frac{1}{n_0} \right),$$

respectively. Also, the versions of $\hat{Q}_1$, $\hat{Q}_2$, and $\hat{Q}_3$ in equations (3.36) through (3.41), for design $D_{N\alpha}$, are

$$\mathbf{Q}_1 = \frac{1}{18}\mathbf{Q}_A - \frac{20}{648}\mathbf{Q}_B + \frac{100}{1755}\mathbf{Q}_C + \frac{127}{1755}\mathbf{R},$$

$$\mathbf{Q}_2 = \frac{1}{18}\mathbf{Q}_A - \frac{37}{648}\mathbf{Q}_B - \frac{1}{12}\mathbf{R},$$

$$\mathbf{Q}_3 = \frac{1}{36}\mathbf{Q}_A - \frac{1}{12}\mathbf{R},$$

where

$$\mathbf{Q}_A = \mathbf{I}_3 \otimes \mathbf{C}^* - \frac{1}{12}\mathbf{J}_{36}$$

$$\mathbf{Q}_B = \mathbf{I}_3 \otimes \mathbf{D}^* - \mathbf{I}_3 \otimes \mathbf{C}^*,$$

$$\mathbf{Q}_C = \mathbf{I}_3 \otimes \mathbf{E}^* - \mathbf{I}_3 \otimes \mathbf{D}^*,$$

$$\mathbf{R} = \mathbf{I}_3 \otimes \mathbf{F}^*.$$

and $\mathbf{C}^* = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \end{bmatrix}$, $\mathbf{D}^* = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \end{bmatrix}$, $\mathbf{E}^* = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \end{bmatrix}$ and $\mathbf{F}^* = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \end{bmatrix}$. Again, the matrices $\mathbf{Q}_A$, $\mathbf{Q}_B$, $\mathbf{Q}_C$ and $\mathbf{R}$ are obtained by using equations (3.6) through (3.8) with design D_{full} . By applying Theorem 3.4.1 with the matrices $\mathbf{Q}_A$, $\mathbf{Q}_B$, $\mathbf{Q}_C$, Σ_A , Σ_B and Σ_C , the distribution of $W_{A,1}$, $W_{B,1,2,3}$ and $W_{C,1,2,3}$ for the design D_{full} are obtained.

The quantiles of $\hat{W}_{A,1}$, $\hat{W}_{B,1,2,3}$ and $\hat{W}_{C,1,2,3}$ for designs D_{full} , D_{eq} and D_{alt} are computed using Davoodi (2002) algorithm. For each design, values of maximum and minimum quantiles of the scaled ANOVA estimators, which are defined in (3.6) through (3.8), are obtained over the region of the parameter space, namely $\delta_{\mathbf{q}}$ for each of several values of p . The region is chosen according to the one used by Jones et al. (2002) so that we can compare our findings to theirs. It is given by

$$\delta_{\mathbf{q}} = \{(\eta_1, \eta_2, \eta_3) \mid 1 \leq \eta_1 \leq 3, 1 \leq \eta_2 \leq 3, 1 \leq \eta_3 \leq 3\} \quad (3.9)$$

More specifically for a given p , a quantile value $q_{1-\alpha}(p, \mathbf{q})$ is computed using the following values of $\mathbf{q} = (\eta_1, \eta_2, \eta_3)$: $q_1 = 1/2(\eta_1 = 1)(\eta_2 = 1)(\eta_3 = 1)$. The maximum and

Table 3.10: Minimum and maximum possible values of $W_{L,2}$ for selected values of p for designs $D_{6,2}$, $D_{6,3}$ and $D_{6,4}$.

p	$D_{6,2}$		$D_{6,3}$		$D_{6,4}$	
	min.	max.	min.	max.	min.	max.
0.01	11.662	4.607	10.477	5.609	12.262	5.131
0.05	1.666	6.662	4.579	6.771	1.756	6.760
0.10	-1.951	8.160	-4.104	6.563	1.465	6.466
0.20	-4.693	6.655	-6.661	6.540	1.739	6.607
0.30	-6.424	5.596	2.147	6.671	1.661	6.756
0.40	-5.975	5.660	-6.561	6.764	1.363	6.650
0.50	-5.173	5.676	6.661	6.674	0.839	5.666
0.60	1.043	3.666	6.661	3.663	1.666	3.666
0.70	1.566	3.666	6.666	4.675	1.666	3.666
0.80	1.666	6.166	6.666	6.666	1.666	6.767
0.90	5.666	6.767	6.766	6.676	1.676	6.767
0.95	6.566	11.676	6.666	11.666	1.666	16.676
0.99	5.666	16.766	6.666	16.666	1.666	16.766

minimums of the resulting possible values, $Q_{L,2}^{(p)}(x)$ and $Q_{L,2}^{(p)}(x)$, for $W_{L,2}$, $Q_{L,2}^{(p)}(x)$ and $Q_{L,2}^{(p)}(x)$ for $W_{D_{6,2,2}}$ and $Q_{L,2}^{(p)}(x)$ and $Q_{L,2}^{(p)}(x)$, for $W_{D_{6,2,3}}$ are given in Tables 3.10, 3.11 and 3.12 respectively. The QDCs for $D_{6,2}$, $D_{6,3}$ and $D_{6,4}$ are constructed on the basis of these tables. Figure 3.14 shows the QDCs of $W_{D_{6,2}}$, whereas Figure 3.15 shows the QDCs of $W_{D_{6,3,2}}$ and Figure 3.16 shows the QDCs of $W_{D_{6,3,3}}$. The following observations can be drawn from these figures:

- (a) The variability in the quantile values of $W_{L,2}$ with respect to p for designs $D_{6,2}$ and $D_{6,3}$ or for designs $D_{6,2}$ and $D_{6,4}$ are quite similar [see Figure 3.14(a,c)]. The variability for design $D_{6,4}$ is smaller than that for design $D_{6,2}$ but the

Table 3.11: Minimum and maximum quantile values of $\hat{H}_{\text{max}}$ for selected values of p for designs D_{int} , D_{eq} and D_{ext} .

p	D_{int}		D_{eq}		D_{ext}	
	min.	max.	min.	max.	min.	max.
0.01	7.175	0.000	-1.025	-0.143	11.475	0.120
0.05	-5.054	-0.363	0.040	0.290	-4.375	0.214
0.10	-0.906	-0.201	-0.040	0.100	-4.543	0.407
0.25	-0.546	-0.420	-0.294	0.501	-0.545	0.501
0.50	-1.097	-0.669	-1.097	1.097	1.090	0.750
0.60	-0.979	-0.758	-0.291	0.904	-0.175	0.920
0.75	0.507	-0.901	-0.053	0.998	0.113	1.100
0.80	1.010	1.104	1.004	0.999	1.001	1.094
0.90	1.000	0.000	1.001	0.001	1.004	0.000
0.95	1.000	-1.101	1.000	-1.000	1.004	-0.000
0.99	1.000	11.547	1.000	11.540	0.000	-11.502

Table 3.11: Minimum and maximum specific values of $\hat{W}_{p(a),k}$ for selected values of p for designs D_{bal} , D_{eq} and D_{hor} .

p	D_{bal}		D_{eq}		D_{hor}	
	min.	max.	min.	max.	min.	max.
-0.99	-0.239	0.203	-0.435	0.197	-1.194	1.194
-0.95	-0.444	0.199	-0.559	0.195	-1.664	1.667
-0.90	-0.594	0.194	-0.699	0.193	-2.154	1.674
-0.85	-0.685	0.197	-0.795	0.194	-2.595	0.195
-0.80	-0.767	0.193	-0.873	0.194	-3.095	0.694
-0.75	-0.797	0.193	-0.913	0.194	-3.495	0.793
-0.70	-0.808	0.193	-0.933	0.195	-3.905	0.793
-0.65	-1.006	1.195	-1.053	1.193	-1.065	1.195
-0.60	-1.133	1.193	-1.264	1.193	-1.133	1.193
-0.55	-1.364	1.193	-1.374	1.193	-1.355	1.193
-0.50	-1.633	1.193	-1.633	1.193	-1.764	1.193
-0.45	-1.677	0.693	-1.693	0.693	-2.065	0.693
-0.40	-0.693	0.697	-0.693	0.693	-0.693	0.697

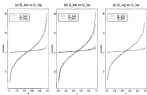


Figure 3.18: Q-Q plots of $W_{n,h}$ for the Glee designs, $G_{1,0.05}$, $G_{1,0.1}$ and $G_{1,0.2}$.

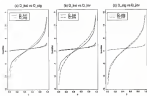


Figure 5.31: Q-Qs of $H_{p,0.05}$ for the three designs, $D_{0.05}$, $D_{0.10}$ and $D_{0.20}$.

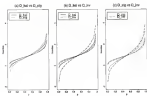


Figure 3.11: QDCs of $W_{q,avg}$ for the three designs, D_{300} , D_{300} and D_{300} .

difference is not apparent. Thus, a slightly more stable ANOVA estimate of σ_e^2 can be obtained with D_{low} than with D_{high} [see Figure 3.14(b)].

- (b) The variability in the quantile value of $W_{D_{low},A}$ with respect to η for design D_{low} is smaller than that for design D_{high} and D_{low} [see Figure 3.14(a,b)]. Design D_{high} shows smaller variability in design D_{low} [see Figure 3.14(a)]. Hence, a more precise estimate of $\sigma_{D_{low}}^2$ is obtained with D_{low} than with either D_{high} or D_{low} .
- (c) The variability in the quantile value of $W_{D_{high},A}$ with respect to η is larger with designs D_{high} and D_{low} than with design D_{high} [see Figure 3.14(a,b)]. The smaller degree of variability with D_{high} than with D_{low} is more apparent here [see Figure 3.14(c)]. Also, D_{high} provides smaller variability than D_{low} [see Figure 3.14(c)]. The degree of variability in the quantile value of $W_{D_{high},A}$ is also larger for D_{low} than for D_{low} [see Figure 3.14(b)]. Hence, a more stable estimate of $\sigma_{D_{high}}^2$ is obtained with D_{high} than with either D_{high} or D_{low} . Also, a more stable estimate is obtained with D_{high} than with D_{low} .

In summary, the use of design D_{high} or D_{low} is slightly preferred over design D_{low} only when the ANOVA estimate of σ_e^2 is of interest. This is expected since D_{high} and D_{low} have larger degrees of freedom for factor A than D_{high} . However, the amount of gain using D_{high} or D_{low} over D_{low} is not large enough to justify being inconsistent if it is more important for us, experimenters, to obtain a stable ANOVA estimate of σ_e^2 as compared to $\sigma_{D_{low}}^2$ and $\sigma_{D_{high}}^2$, then it is recommended to use D_{low} rather than D_{high} . Otherwise, the balanced design is preferred for estimating $\sigma_{D_{low}}^2$ or $\sigma_{D_{high}}^2$. The preference of D_{high} over D_{low} occurs only when $\sigma_{D_{high}}^2$ is to be estimated. Otherwise, there is no distinct difference between the two designs.

3.4.3. Comparison of designs using ML estimation

Let us denote the ML estimator of σ_{μ}^2 , $\sigma_{\delta_{ijk}}^2$ and $\sigma_{\epsilon_{ijkl}}^2$ by $\hat{\sigma}_{\mu}^2$, $\hat{\sigma}_{\delta_{ijk}}^2$ and $\hat{\sigma}_{\epsilon_{ijkl}}^2$, respectively. Let us also denote the corresponding scaled ML estimator by $W_{\mu} = \frac{\hat{\sigma}_{\mu}^2}{\hat{\sigma}_{\epsilon_{ijkl}}^2}$, $W_{\delta_{ijk}} = \frac{\hat{\sigma}_{\delta_{ijk}}^2}{\hat{\sigma}_{\epsilon_{ijkl}}^2}$, and $W_{\epsilon_{ijkl}} = \frac{\hat{\sigma}_{\epsilon_{ijkl}}^2}{\hat{\sigma}_{\epsilon_{ijkl}}^2}$. The EQD(s) of W_{μ} , $W_{\delta_{ijk}}$ and $W_{\epsilon_{ijkl}}$ for each design are obtained using the following method of the Monte-Carlo simulation.

- (a) Without any loss of generality, we assume that $\mu = 0$ in model (2.83). We then have

$$y_{ijk\ell} = \eta_i + \delta_{ijk} + \epsilon_{ijk\ell} + \epsilon_{ijk\ell} \quad (3.100)$$

$i = 1, 2, \dots, n$; $j = 1, 2, \dots, h_i$; $k = 1, 2, \dots, r_{ij}$ and $\ell = 1, 2, \dots, r_{ijk}$. We

recall that $\eta_i = \frac{\sigma_{\mu}^2}{\sigma_{\epsilon_{ijkl}}^2}$, $\delta_{ijk} = \frac{\sigma_{\delta_{ijk}}^2}{\sigma_{\epsilon_{ijkl}}^2}$ and $\epsilon_{ijk\ell} = \frac{\sigma_{\epsilon_{ijkl}}^2}{\sigma_{\epsilon_{ijkl}}^2}$. Hence, $\eta_i \sim N(0, \sigma_{\mu}^2)$, $\delta_{ijk} \sim N(0, \sigma_{\delta_{ijk}}^2)$, $\epsilon_{ijk\ell} \sim N(0, \sigma_{\epsilon_{ijkl}}^2)$, and $\epsilon_{ijk\ell} \sim N(0, \sigma_{\epsilon_{ijkl}}^2)$. Let R denote a region in the parameter space of the quadruple $(\eta_i, \delta_{ijk}, \epsilon_{ijk\ell})$, where $\eta = (\eta_i, \delta_{ijk}, \epsilon_{ijk\ell})$.

- (b) For a given $(\eta, \delta_{ijk}, \epsilon_{ijk\ell})$ in R and a given design D , a random vector y is generated on the basis of model (3.100), i.e. random number, which we denote by $y_{ijk\ell}$, is generated from $N(0, \sigma_{\epsilon_{ijkl}}^2)$. Independently from $y_{ijk\ell}$, $\epsilon_{ijk\ell}$ independent random numbers, which we denote by $\epsilon_{ijk\ell}, \epsilon_{ijk\ell}, \dots, \epsilon_{ijk\ell}$, are generated from $N(0, \sigma_{\epsilon_{ijkl}}^2)$. Independently from $y_{ijk\ell}$ and $\epsilon_{ijk\ell}, \epsilon_{ijk\ell}$ independent random numbers, which we denote by $\epsilon_{ijk\ell}, \epsilon_{ijk\ell}, \dots, \epsilon_{ijk\ell}$, are generated from $N(0, \sigma_{\epsilon_{ijkl}}^2)$. In addition, δ_{ijk} independent random numbers, which we denote by $\delta_{ijk}, \delta_{ijk}, \dots, \delta_{ijk}$, are generated from $N(0, \sigma_{\delta_{ijk}}^2)$ independently from $y_{ijk\ell}, \epsilon_{ijk\ell}$ and $\epsilon_{ijk\ell}$. This process produces a data vector, $(y_{ijk\ell} + \delta_{ijk} + \epsilon_{ijk\ell} + \epsilon_{ijk\ell}, y_{ijk\ell} + \delta_{ijk} + \epsilon_{ijk\ell} + \epsilon_{ijk\ell}, \dots, y_{ijk\ell} + \delta_{ijk} + \epsilon_{ijk\ell} + \epsilon_{ijk\ell})'$, which represents response value from an r^{th} first-stage testing group ($i = 1, 2, \dots, n$). By repeating this process independently a times and combining

the results, we obtain a vector $\mathbf{g}$ consisting of $\sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N g_{ijk} = N$ observations.

- (c) Using the response vector $\mathbf{g}$ obtained in step (b), ML estimation, σ_{ML}^2 , $\sigma_{\text{ML}|\text{ML}}^2$ and $\sigma_{\text{ML}|\text{ML}}^2$ are computed. Values of $\bar{W}_{\text{ML}}$, $\bar{W}_{\text{ML}|\text{ML}}$, $\bar{W}_{\text{ML}|\text{ML}}$ and $\bar{W}_{\text{ML}|\text{ML}}$ are then obtained.
- (d) By repeating steps (b) and (c) a sufficient number of times, we can obtain expected quantities of $\bar{W}_{\text{ML}}$, $\bar{W}_{\text{ML}|\text{ML}}$ and $\bar{W}_{\text{ML}|\text{ML}}$ for selected values of p for a given $\{\mathbf{g}, \sigma_{\text{ML}}^2\}$. We denote the p^{th} expected quantity thus obtained by $\bar{E}_p^{\text{ML}}(\mathbf{g}, \sigma_{\text{ML}}^2)$ for $\bar{W}_{\text{ML}}$, $\bar{E}_p^{\text{ML}}(\mathbf{g}, \sigma_{\text{ML}}^2)$ for $\bar{W}_{\text{ML}|\text{ML}}$ and $\bar{E}_p^{\text{ML}}(\mathbf{g}, \sigma_{\text{ML}}^2)$ for $\bar{W}_{\text{ML}|\text{ML}}$.
- (e) Steps (b), (c) and (d) are repeated several times corresponding to different values of $\{\mathbf{g}, \sigma_{\text{ML}}^2\}$ selected from the region $\mathcal{R}$. The set of such selected quadruplets is denoted by $\mathcal{B}$. Thus $\mathcal{B} \subset \mathcal{R}$.
- (f) The maximum and minimum of the p^{th} expected quantity values over the set $\mathcal{B}$ in step (e) are obtained for each of $\bar{W}_{\text{ML}}$, $\bar{W}_{\text{ML}|\text{ML}}$ and $\bar{W}_{\text{ML}|\text{ML}}$. We then have, for $\bar{W}_{\text{ML}}$

$$\hat{Q}_{\text{ML}}^{\text{ML}}(p) = \max_{\{\mathbf{g}, \sigma_{\text{ML}}^2\} \in \mathcal{B}} \bar{E}_p^{\text{ML}}(\mathbf{g}, \sigma_{\text{ML}}^2), \quad (2.104)$$

$$\hat{Q}_{\text{ML}}^{\text{ML}}(p) = \min_{\{\mathbf{g}, \sigma_{\text{ML}}^2\} \in \mathcal{B}} \bar{E}_p^{\text{ML}}(\mathbf{g}, \sigma_{\text{ML}}^2), \quad (2.105)$$

for $\bar{W}_{\text{ML}|\text{ML}}$

$$\hat{Q}_{\text{ML}|\text{ML}}^{\text{ML}}(p) = \max_{\{\mathbf{g}, \sigma_{\text{ML}}^2\} \in \mathcal{B}} \bar{E}_p^{\text{ML}}(\mathbf{g}, \sigma_{\text{ML}}^2), \quad (2.106)$$

$$\hat{Q}_{\text{ML}|\text{ML}}^{\text{ML}}(p) = \min_{\{\mathbf{g}, \sigma_{\text{ML}}^2\} \in \mathcal{B}} \bar{E}_p^{\text{ML}}(\mathbf{g}, \sigma_{\text{ML}}^2), \quad (2.107)$$

and, for $\hat{W}_{\text{design}}$,

$$\hat{Q}_{\text{DES}}^{\text{max}}(x) = \max_{\mathbf{q} \in \mathcal{Q}_{\text{DES}}} \hat{Q}_{\text{DES}}^{\text{max}}(\mathbf{q}, x, x), \quad (3.106)$$

$$\hat{Q}_{\text{DES}}^{\text{min}}(x) = \min_{\mathbf{q} \in \mathcal{Q}_{\text{DES}}} \hat{Q}_{\text{DES}}^{\text{min}}(\mathbf{q}, x, x). \quad (3.107)$$

- (g) The maximum and minimum quartile values obtained from (3.104) to (3.106) are plotted against x to obtain the SQPGs of $\hat{W}_{\text{DES}}$, $\hat{W}_{\text{DES}}^{\text{max}}$ and $\hat{W}_{\text{DES}}^{\text{min}}$, respectively.

Let us now illustrate the application of the above procedure using designs D_{DES} , $D_{\text{DES}}^{\text{max}}$ and $D_{\text{DES}}^{\text{min}}$. The region R is chosen as

$$R = \{(\mathbf{q}, x_1^T) \mid 1 \leq q_1 \leq 9, 1 \leq q_2 \leq 9, 1 \leq q_3 \leq 9, 0.1 \leq x_1^T \leq 10\} \quad (3.107)$$

The set R mentioned in step (g) consists of 810 points selected from R such that

$$R = \{(\mathbf{q}, x_1^T) \mid q_1 = 1(2)9, q_2 = 1(2)9, q_3 = 1(2)9, x_1^T = 0.1, 0.4, 1, 2, 10\}$$

A SAS Macro (SAS, 1990) was written to compute $\hat{Q}_{\text{DES}}^{\text{max}}(\mathbf{q}, x, x)$, $\hat{Q}_{\text{DES}}^{\text{min}}(\mathbf{q}, x, x)$ and $\hat{Q}_{\text{DES}}^{\text{max}}(\mathbf{q}, x, x)$ values. The PROC NODD procedure in SAS (1997) was used to obtain the estimates of variance components in the full Matrix program. A total of 1000 replications in step (c) were made to calculate required quartile values for each design and each quadruple of $(\mathbf{q}, x_1^T)$ in the region R . Tables 3.13, 3.14, and 3.15 give these maxima and minima which are defined in (3.104) through (3.106) for designs D_{DES} , $D_{\text{DES}}^{\text{max}}$ and $D_{\text{DES}}^{\text{min}}$.

Using these tables, the SQPGs of $\hat{W}_{\text{DES}}$, $\hat{W}_{\text{DES}}^{\text{max}}$ and $\hat{W}_{\text{DES}}^{\text{min}}$ are plotted, as shown in Figures 3.17–3.19, respectively. In each figure, comparisons for each pair of designs can be made. From these figures we note similar results as for LQGNs estimation. The unbalanced nested designs, D_{DES} and $D_{\text{DES}}^{\text{min}}$, provide better

Table 5.15: Minimum and maximum empirical quantile values of $\hat{R}_{\text{max}}$ for selected values of p for changes D_{low} , D_{up} and D_{low} .

p	D_{low}		D_{up}		D_{low}	
	min.	max.	min.	max.	min.	max.
0.05	0.000	0.002	0.000	0.173	0.000	0.106
0.10	0.000	0.170	0.000	0.210	0.000	0.401
0.15	0.000	0.177	0.000	0.430	0.000	0.470
0.20	0.000	0.440	0.000	0.730	0.000	0.512
0.25	0.000	0.170	0.000	0.700	0.000	0.710
0.30	0.000	0.740	0.000	0.971	0.000	0.881
0.35	0.280	1.000	0.340	1.110	0.210	1.100
0.40	0.320	1.300	0.660	1.501	1.010	0.607
0.45	1.100	6.711	1.740	3.601	1.364	0.500
0.50	1.440	5.601	1.300	5.601	1.377	0.760
0.55	1.107	5.114	1.040	9.300	1.124	7.000
0.60	1.004	10.540	1.000	10.011	1.700	9.000
0.65	1.000	10.260	0.000	14.000	0.110	14.000

Table 2.14. Minimum and maximum empirical quantile values of $W_{D_{\text{max}}}$ for selected values of γ for designs D_{bal} , D_{bal_2} , and D_{max} .

γ	D_{bal}		D_{bal_2}		D_{max}	
	min.	max.	min.	max.	min.	max.
0.01	0.000	0.000	0.000	0.150	0.000	1.300
0.05	0.000	0.145	0.000	0.100	0.000	0.950
0.20	0.000	0.107	0.000	0.400	0.000	0.650
0.50	0.000	0.090	0.000	0.375	0.000	0.604
0.70	0.000	0.102	0.000	0.400	0.000	0.750
0.80	0.000	0.700	0.000	0.420	0.000	1.000
0.90	0.000	0.600	0.100	1.000	0.170	1.100
0.95	0.000	1.000	0.000	0.500	0.000	0.600
0.99	1.000	1.000	0.000	0.000	0.000	0.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000
1.01	1.000	1.000	1.000	1.000	1.000	1.000
1.05	1.000	1.000	1.000	1.000	1.000	1.000
1.10	1.000	1.000	1.000	1.000	1.000	1.000
1.20	1.000	1.000	1.000	1.000	1.000	1.000
1.50	1.000	1.000	1.000	1.000	1.000	1.000
2.00	1.000	1.000	1.000	1.000	1.000	1.000

ML estimate of σ_{true}^2 from the balanced nested design, D_{bal} , in terms of smaller variability in the quantile values for $\mathcal{W}_{\text{true}}$ [see Figure 3.17(a,b)]. However, in estimating σ_{true}^2 or σ_{true}^2 , the components of D_{bal} in D_{bal} and D_{unb} is equal [see Figure 3.18(a,b) and Figure 3.19(a,b)]. Design D_{bal} is preferred over D_{unb} only when the ML estimation of σ_{true}^2 is considered [see Figure 3.19(c)]. Otherwise, there is no clear preference between them [see Figure 3.18(c) and 3.19(c)].

In conclusion, the staggered nested design does not provide an overall gain in terms of its variability in the quantile values for the nested estimator in comparison to the balanced nested design.

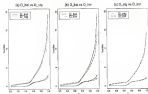


FIGURE 3.17 BQQCs of $\mathcal{W}_{\text{true}}$ for the three designs, D_{bal} , D_{unb} , and D_{mf} .

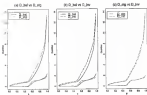


Figure 3.15. ECOCs of $M_{(1), (2)}$ for the three designs, $D_{(1), (2)}$, $D_{(1), (2)}$, and $D_{(1), (2)}$.

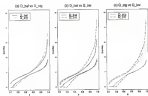


Figure 3.13: BQD-G of $W_{L,upper}$ for the three designs, $D_{L,med}$, $D_{L,reg}$ and $D_{L,low}$.

3.4.3. Comparison of ANOVA and ML estimation

Comparisons of ANOVA estimation with that of ML with regard to σ_{η}^2 , $\sigma_{\epsilon_{ijk}}^2$ and $\sigma_{\epsilon_{ijm}}^2$ are made in Figures 3.30, 3.31, and 3.32. Again, note that the comparison is biased because the regions Ω_{η} in (3.18) and $\tilde{R}$ in (3.167) are not identical. Aside from the possibility of having negative estimation of the variance components, the ANOVA estimation are comparable to the ML estimation as the quantile values of σ_{η}^2 and $\sigma_{\epsilon_{ijk}}^2$ [see Figures 3.30 and 3.31]. This holds true for all three designs. It is noted that the frequency of getting negative ANOVA estimates of σ_{η}^2 and $\sigma_{\epsilon_{ijk}}^2$ is very high. The variability in the quantile values with respect to η for W_{balanced} is smaller than that for $W_{\text{unbalanced}}$ with respect to (η, σ_{η}^2) , except for the possibility of getting negative values of variance components estimation [see Figure 3.32]. This holds true over the entire range of μ .

3.4. Conclusion

The QDCs associated with ANOVA estimation, and the RLQDCs for ML estimation, provide a powerful graphical tool for comparing designs for estimating variance components. The dependency of the designs on the unknown values of the variance components was circumvented by the consideration of dispersion of the quantiles of a given estimator over a certain region of the parameter space. This makes it possible to compare designs uniformly with respect to the unknown parameters.

We compare balanced designs with several unbalanced designs for various models, namely the one-way random model, the two-way cross classification without interaction random model, and the three-fold nested random model. As was stated in Chapter 2, several studies have shown that, using various single-sided criteria, the

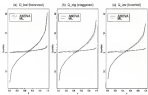


Figure 3-30: QDOs of $\hat{W}_{opt}$ and WQDOs of $\hat{W}_{opt}$ for the three designs: D_{opt} , D_{opt} and D_{opt} .

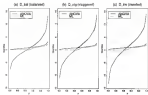


Figure 3.10: QEQs of P_{out} and EQEQs of P_{out} for the three designs: D_{out} , D_{out} and D_{out} .

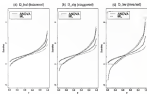


Figure 2.32 QEDs of W_{optimal} and EQEDs of K_{optimal} for the three designs, D_{opt} , D_{rigid} , and D_{flexible} .

balanced design is not always favorable to estimate variance components over unbalanced designs. Three studies were based on the large sample properties of ML estimation (Blatt and Anderson, 1978; Blatt, Anderson, and Tinselman, 1982) or on the empirical distribution of ANOVA estimates (Lange and Nelson, 1988; Lange et al., 1989). Based on our study to assess the overall quality of the estimation ability through the QGCs or the EQGCs, we can determine which unbalanced design provides better estimates than the balanced design. By a better design it is meant a design which provides smaller variability in the quantile values of the scaled estimates, or which provides a larger degree of uniformity of such values. The QGCs or the EQGCs allow us to compare designs for each of the variance components. For example, as we see in Section 2.4, the choice of a nested design can be decided by which variance component is of most interest to an experimenter.

The comparison of ANOVA estimation with ML estimation is also valid. Even though this comparison is limited since the regions of the parameter space are not identical, it has shown that the ANOVA estimation provides estimates comparable to ML estimates except for the possibility of having negative ANOVA estimates. That is an interesting result because it is based on small sample properties of ML estimators, rather than on an asymptotic behavior. Only for some unbalanced designs for particular models, for example a design D_{12} for the one-way random model, and designs D_{22} and D_{23} for the two-way cross-classification random model, does ML provide usually better estimates than ANOVA (see Figures 2.8(c) and 2.11(b,c)).

CHAPTER 4 INVESTIGATION OF THE EFFECT OF IMBALANCE

4.1 The Probability of a Negative ANOVA Estimate

4.1.1 Introduction

It is commonly known that the analysis of variance (ANOVA) method of estimation of variance components does not preclude the possibility of having negative estimates. This problem has been discussed by several authors, particularly in the case of the one-way random model. See, for example, Kelly and Mathew (1983, 1984), Leone et al. (1987), Pothakamuri and Jais (1987), Singh (1989), Tiao and Wong (1976), Venkatesh (1983) and Wong (1987), to name just a few.

The coefficient of an ANOVA estimate, $\hat{\mu}^2$, of a variance component, σ^2 , can be measured by the size of the probability $P(\hat{\mu}^2 < 0)$. Of particular interest in this context is the ability to study the behavior of $P(\hat{\mu}^2 < 0)$, and to identify the factors that influence such a behavior. Some of these factors include the type of design used and the true values of the variance components for the model under consideration. The task of undertaking this sort of investigation can be quite daunting when the data set is unbalanced. For example, it is very difficult to determine the pattern of imbalance that adversely affects the value of $P(\hat{\mu}^2 < 0)$ because of the dependency of the probability on the model's variance components. The difficulty is compounded by the non-uniqueness of ANOVA estimates due to the availability of several ANOVA tables for the same unbalanced data. Furthermore, the same is equally true in an ANOVA

table set, in general, neither independent nor have the de-symmetrization property as in the case with balanced data under the usual ANCOVA assumptions.

The purpose of Section 4.1 is to study the behavior of $P(\hat{\sigma}_e^2 < 0)$ by modeling its value in terms of factors which are considered to influence such a behavior. Generalized linear models (GLMs) techniques are used for this purpose. The proposed modeling approach provides a better insight into the effects of data imbalance on the quality of ANCOVA estimation of variance components. In Section 4.1.1, the probability of getting a negative ANCOVA estimate with one-way random model is formulated. A method for generating designs having a specified degree of imbalance for the one-way model is mentioned in Section 4.1.2. In Section 4.1.3, this probability is modeled separately using generalized linear models techniques. The predictive capability of the fitted model is examined in Section 4.1.4, and concluding remarks are given in Section 4.1.5.

4.1.1. The probability of a negative ANCOVA estimate for the one-way model

Consider the unbalanced one-way random model (3.1) and its vector form in (3.4). The variance-covariance matrix of the vector of observed values (which is denoted by $\mathbf{Z}$), is shown in (3.5). With model (3.4), the probability of a negative ANCOVA estimate of σ_e^2 , which we denote by $\sigma_{\hat{e}}^2$, can be found using Corollary 3.1.1 namely

$$P(\sigma_{\hat{e}}^2 < 0) = P\left(\frac{1}{J-1} \sum_{j=1}^J \lambda_j^2 \mathbf{Z}_{(j)}^T \mathbf{Z}_{(j)} - \frac{1-n}{N-1-J} \mathbf{Z}_{(0)}^T \mathbf{Z}_{(0)} < 0\right), \quad (4.1)$$

where $\lambda_1, \lambda_2, \dots, \lambda_J$ are the distinct positive eigenvalues of $\mathbf{Q}\mathbf{K}^*$ with multiplicity $m_1, m_2, \dots, m_J$, respectively, the matrix $\mathbf{Q}$ was defined in (3.7), and the matrix $\mathbf{K}^*$ is defined as

$$\mathbf{K}^* = \sigma_{\mu_0}^2 \mathbf{J}_{n_0} + \frac{1-n}{J} \mathbf{I}_{n_0}. \quad (4.2)$$

where $\rho = \frac{1}{2\sqrt{1-\phi}}$. Since the left hand side of the inequality inside the probability (4.1) is a linear combination of independent normal distributed variables, the probability of a negative ANOVA estimate of σ_0^2 can be computed for specified value of ρ .

Let us denote a design for model (3.1) by $D = \{\eta_0, \eta_1, \dots, \eta_r\}$. Note that the values of η_i^j 's and the corresponding m_i/n_i , $i = 1, 2, \dots, r$, are determined by the design D and ρ . Moreover, as will be seen in the next section, a design D for model (3.1) can be generated by specifying the total number of observations, N , and the degree of imbalance, δ , when the number of treatments, k , is fixed. The degree of imbalance, δ , is a quantity that measures departure of a design from the completely balanced design, which will be defined in the next section.

The exact value of (4.1) can be obtained by using a computer algorithm given by Garver (1988) as was mentioned in Section 3.1.2. It should be noted that the exact probability value in (4.1) is not an explicit function of the parameters N , ϕ , and ρ . It can, however, be modeled explicitly in terms of N , δ , and ρ . Using this probability as a response variable and the parameters N , δ , and ρ as control variables, we can establish such a relationship in order to study the effect of imbalance on $P(\hat{\sigma}_0^2 < 0)$. The proposed modeling of $P(\hat{\sigma}_0^2 < 0)$ depends on a method for generating designs having a specified degree of imbalance. This will be discussed in the next section.

4.1.1. Generation of designs with a specified degree of imbalance

A design that is a set of η_i values for model (3.1), can be generated for given values of N , δ , and k . To measure the degree of imbalance, Morris and Plous (1981) introduced two measures of imbalance for model (3.1). One of them is given by

$$\delta = \frac{N^2}{k \sum_{i=1}^r n_i^2} \quad (4.2)$$

where $0 < \delta \leq 1$. The smaller the δ , the larger the degree of imbalance. The value 1 is attained when the data set is balanced. Elmeri (1986) introduced a method for

Table 4.1: Number of designs generated for each combination of N , ϕ , and p

N	25			50			75			
ϕ	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	
p	0.2	5	5	7	5	15	17	7	19	25
	0.4	5	5	7	5	15	17	7	19	25
	0.6	5	5	7	5	15	17	7	19	25

generating designs $D = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ having specified values of N and ϕ . This method will be used to model $P(y_i^* \in Q)$ as a response function of N , ϕ , and p .

To demonstrate the development of the aforementioned relationship, let us, for example, consider that $k = 3$ in model (3.1). Thus $D = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Several combinations of N , ϕ , and p are selected according to a 3^3 factorial arrangement. The selected values are: $N = 25, 50, 75$, $\phi = 0.25, 0.50, 0.75$, $p = 0.2, 0.4, 0.6$. For each assignment of the triple (N, ϕ, p) , several designs are generated on the basis of Kiefer's (1966) method. The application of this method was carried out using a computer program written in the S-PLUS language. This program is included in the appendix. Note that the generation of designs requires only the specification of N and ϕ . The generated designs along with the value of p determine the column η -vectors of the matrix $\mathbf{Q}\mathbf{Q}^T$, and hence the probability in (4.1). Table 4.1 lists the number of designs generated for each of the 27 combinations of N , ϕ , and p . Unequal numbers of designs for each combination of the levels of N and ϕ are generated since the availability of possible designs varies according to the values of N and ϕ . The total number of designs thus obtained is 105.

4.1.4. Numerical analysis of $P(x_i^* < 0)$

For each selection of (N, δ, ρ) in Table 4.1, several designs are generated. Some of these designs are listed in Table 4.2. In this table, δ_i denotes the actual value of δ for a δ -generated design. For each design, the constant eigenvalues of QE^T are obtained where Q and E^T are defined in (3.7) and (3.9), respectively. The corresponding value of $P(x_i^* < 0)$ in (3.1) is computed using Davies' (1980) algorithm. The constant eigenvalues of QE^T and the values of $P(x_i^* < 0)$ corresponding to some of the generated designs are given in the Table 4.2. Since several designs are generated for each selection of N and δ , the values of $P(x_i^* < 0)$ corresponding to the triple (N, δ, ρ) represent replicated response values with N , δ , and ρ acting as fixed variables. Let us therefore denote a value of $P(x_i^* < 0)$ by p .

The data in Table 4.2 can now be used to model the probability of a response x_i^* against N , δ and ρ . Given the values of the response y_i being a probability, it would be appropriate to use generalized linear models for this purpose. The logistic regression fitting is appropriate since the response variable and its true probability act as binomial probability obtained as the ratio of the number of successes to the number of trials.

Let us consider the transformation

$$w_{i\alpha} = \left[\frac{y_i}{1 - y_i} \right]^{1/\alpha}, \quad \alpha \geq 1 \quad (4.4)$$

which maps the interval $[0, 1]$ onto $[0, \infty)$. Since several replications are available on y_i and hence on $w_{i\alpha}$ for each triple (N, δ, ρ) in Table 4.2, the mean $\bar{w}_{i\alpha}$ and sample variance $s_{i\alpha}^2$ of $w_{i\alpha}$ for the i^{th} triple can be computed. Plotting $\bar{w}_{i\alpha}$ against $s_{i\alpha}^2$ or against $\bar{w}_{i\alpha}$ for the various values of α , $\alpha = 1, 2, \dots, 10$, may suggest a possible linear relationship between the mean and variance of $w_{i\alpha}$ or between the mean and standard deviation of $w_{i\alpha}$.

Table 4.3: Generated designs for the one-way model with $\text{dof}(\hat{\eta})$ denotes the actual value of η for a ϕ -generated design.

N	ϕ	r	n_0	n_1	n_2	n_3	n_4	n_5	d_n
25	0.25	0.2	100	1	1	0	1	1	0.2091
25	0.25	0.2	1	100	1	0	0	0	0.2099
25	0.25	0.2	0	10	1	1	1	0	0.2061
25	0.25	0.4	1	1	0	10	1	0	0.2061
25	0.25	0.4	1	100	0	1	1	0	0.2091
25	0.25	0.4	0	1	0	10	1	0	0.2099
25	0.25	0.6	0	1	100	1	1	0	0.2071
50	0.50	0.4	100	1	0	7	10	0	0.5040
50	0.50	0.4	0	0	0	1	100	0	0.5040
50	0.50	0.4	100	4	0	0	7	0	0.5054
50	0.50	0.4	0	7	100	0	0	0	0.5050
50	0.50	0.4	101	100	0	4	0	0	0.5061
50	0.50	0.4	127	0	100	7	1	0	0.5040
75	0.75	0.6	100	0	100	101	10	0	0.7404
75	0.75	0.6	4	100	100	10	100	0	0.7410
75	0.75	0.6	100	100	101	7	100	0	0.7409
75	0.75	0.6	100	0	0	100	100	0	0.7409
75	0.75	0.6	100	100	100	0	100	0	0.7409
75	0.75	0.6	0	100	100	100	100	0	0.7409

Table 4.5: The norm algorithm of **QIE** and the exact value of $P(\mathcal{C})$ ($\epsilon = 0$) obtained by using Chavira (1993) algorithm for each of the generated designs in Table 4.2

m_1	m_2	m_3	m_4	m_5	A_1^*	A_2^*	A_3^*	A_4^*	ϵ	$P(\mathcal{C})$ ($\epsilon = 0$)
10	1	1	0	1	0.1737	1.5463	1.0000	0.0000	0.2	0.01319
1	10	1	0	0	0.0000	0.0000	0.0000	0.0000	0.2	0.00000
0	0	1	1	1	0.0000	0.0000	0.0000	0.0000	0.2	0.00000
1	1	0	10	1	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
1	10	0	1	1	0.0137	0.0064	0.0000	0.0000	0.4	0.00079
1	1	0	10	1	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
1	1	10	1	1	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
⋮										
⋮										
⋮										
10	1	0	0	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
0	0	0	1	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
10	0	0	0	0	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
0	0	10	0	0	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
10	10	0	0	0	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
10	0	10	0	0	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
⋮										
⋮										
⋮										
10	0	0	10	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
0	10	10	10	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
10	10	0	0	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
0	10	0	0	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
0	10	10	0	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000
0	10	10	10	10	0.0000	0.0000	0.0000	0.0000	0.4	0.00000

On the basis of the results in Table 4.3, the coefficient of determination, R^2 , for a simple linear regression model without an intercept using W_{tot} and σ_{tot}^2 is 0.75 for $m = 1$, 0.75 for $m = 2$, and 0.66 for $m = 3$ whereas that of using W_{tot} and σ_{tot} is 0.71 for $m = 1$, 0.50 for $m = 2$, and 0.58 for $m = 3$. This indicates a strong linear relationship between the mean and standard deviation of σ_{tot} and therefore $m = 2$ is selected. However, since y is usually not large, values of σ_{tot} are, for the most part, close to zero. We therefore multiply σ_{tot} by 10^3 . Now let w be defined as

$$w = 10^3 \left[\frac{P(\sigma_{\text{tot}}^2 < 0)}{1 - P(\sigma_{\text{tot}}^2 < 0)} \right]^{\frac{1}{2}} \quad (4.6)$$

With this transformation, we can assume that the coefficient of variation of w is constant. That is, σ_w is proportional to μ_w (σ_w or μ_w), where μ_w and σ_w are the mean and standard deviation of w , respectively. In this case, with a logarithmic transformation as a variance-stabilizing transformation, we can use ordinary least squares method of estimation in order to obtain estimates of parameters, but the intercept is then biased (McCullagh and Nelder, 1989, pp. 285–288). On the other hand, using the log link function to obtain linearity between a response variable and control variables and a quadratic variance function to describe the relationship between mean and variance of w , we can use an iteratively reweighted least squares variance function method in order to obtain estimates of parameters. For the quadratic variance function, it is meant that the variance of w depends only on the mean of w through a quadratic variance function, $V(\mu_w)$, such that $\sigma_w^2 = \psi V(\mu_w)$ where ψ is a constant dispersion parameter and $V(\mu_w) = \mu_w^2$. This estimator method can be used with assuming that w has gamma distribution (see McCullagh and Nelder, 1989, p. 294). Therefore, it is now feasible to choose the gamma distribution as a probability distribution for the values of w , which we now identify as the response variable to be modeled.

In order to model μ_{α} against N , δ , and ρ a link function is needed. This function, which we denote by g , relates μ_{α} to the so-called linear predictor ξ which is represented by a linear model in N , δ , and ρ . Although the suggested canonical link for the gamma distribution is the inverse link, we choose instead a lognormal link, $\xi = \log(\mu_{\alpha})$, where $\log(\mu_{\alpha})$ is the natural logarithm of μ_{α} . The reason for this is that for our example, the inverse link produces undesirable results as will be seen later. Thus, the basic components needed for our generalized linear modelling scheme are

- (a) The response variable y in (4.1) is assumed to have a gamma distribution with a mean μ_{α} and a variance σ_{α}^2 .
- (b) The linear predictor is expressed as $\xi = \mathbf{f}'(\mathbf{x})\boldsymbol{\beta}$ where $\mathbf{x} = (x, \delta, \rho)'$ with $x = \log(N)$, $\boldsymbol{\beta}$ is a vector of unknown parameters, and $\mathbf{f}'(\mathbf{x})$ is a vector function of $\mathbf{x}$ such that $\mathbf{f}'(\mathbf{x})\boldsymbol{\beta}$ is a polynomial in x , δ , and ρ of a certain degree. Note that x is used instead of N since $\log(\delta)$ and ρ have values between 0 and 1. This way, for $N = 25, 50, 75$ (the selected values of N in Table 4.1) x will have the values 3.218, 3.912, 4.327 with a range of 1.1.
- (c) The link function between μ_{α} and ξ is the lognormal link, $\xi = \log(\mu_{\alpha})$. Hence,

$$\mu_{\alpha} = e^{\xi}. \quad (5.6)$$

Let $g(\mathbf{x})$ denote the predicted value of $y = F(\mathbf{x})$, $\mathbf{x} \in \mathcal{C}$ as $\mathbf{x} = (x, \delta, \rho)'$ made the region of experimentation, $\mathcal{C}$. This region is determined by the selected values of N , δ , and ρ in Table 4.1, that is

$$\mathcal{C} = \{(x, \delta, \rho) \mid \log(25) \leq x \leq \log(75), 0.05 \leq \delta \leq 0.15, 0.05 \leq \rho \leq 0.85\} \quad (5.7)$$

Let $\phi(\mathbf{x})$ denote the corresponding predicted value of σ in (4.1). Thus, from (5.5) and (4.1) we have

$$\log(\phi(\mathbf{x})) = \mathbf{f}'(\mathbf{x})\boldsymbol{\beta}$$

that is

$$2 \log(\hat{B}) + 2 \log \left[\frac{p(\hat{x})}{1 - p(\hat{x})} \right] = F(x|\hat{B})$$

where $\hat{B}$ is the maximum likelihood estimate of B . Since

$$p(\hat{x}) = \frac{1}{1 + e^{-2[F(x|\hat{B}) + 1] \log(2)}} \quad (4.6)$$

In order to measure the goodness of fit of model (3.4), we use the scaled deviance or the scaled Pearson's chi-squared statistic (see McCullagh and Nelder (1989) pp. 33–34, Sacks (1997) pp. 183–184). The deviance function is defined to be twice the difference between the maximum achievable log likelihood and that obtained under the fitted model. The latter is achieved with a model that has as many parameters as there are observations (saturated model). The scaled deviance is the deviance divided by the maximum likelihood estimate of the dispersion parameter, which for the gamma distribution is equal to the variance divided by the square of the mean (see Sacks (1997) p. 179). The scaled Pearson's chi-squared statistic is obtained as a similar square. Under certain regularity conditions, the scaled version of these two statistics have the asymptotic chi squared distribution with degrees of freedom equal to the number of observations (Sacks is our example) minus the number of parameters in the fitted model (see Sacks, 1997, p. 183). Note that the deviance function increases as more are deleted from the model. Because of this feature, the deviance is used as computer coded model rather than as an absolute measure of goodness of fit.

Several model forms are considered for the known population $\xi = f(x|\beta)$. These models are listed in Table 4.4. The corresponding deviance, scaled deviance, and ratio of the scaled deviance to the residual degrees of freedom are also given in Table 4.4. The GENMOD procedure in SAS (1987) is used to fit these models on the basis of the logitlink link function: $\xi = \log(\mu_0)$.

From Table 4.4 we note that the complete second degree model has the smallest scaled deviance value of 88.146. This model is therefore chosen to represent the

Table 4.5: The standard errors of parameter estimation and goodness-of-fit statistics for model (4.6)

Subset: Deviance = 208.1403, $\text{df} = 202$ (value/ $\text{df} = 1.0306$)						
Subset: χ^2 value = 206.1246, $\text{df} = 202$ (value/ $\text{df} = 1.0205$)						
variable	parameter	estimate	std. err.	χ^2 stat.	P value	
α	β_0	14.9229	2.2394	-6.6864	0.0000	
	β_1	2.7866	1.1243	2.4803	0.0152	
β	β_2	0.0137	1.2347	0.0010	0.9943	
γ	β_3	1.6389	0.9374	1.7433	0.0861	
$\eta\delta$	β_{41}	-1.2883	0.2871	127.3329	0.0001	
$\eta\epsilon$	β_{42}	-0.7379	0.2667	282.7348	0.0001	
$\delta\epsilon$	β_{43}	0.0092	0.4076	149.9179	0.0001	
α^2	β_{51}	-0.4379	0.1451	9.0135	0.0039	
$\alpha\beta$	β_{52}	0.7649	0.1961	150.4949	0.0001	
β^2	β_{53}	1.0771	0.1640	97.6344	0.0001	

linear predictor in (4.6). In this case, $\mathcal{F}(x_0^T\beta)$ has the form

$$\begin{aligned}\mathcal{F}(x_0^T\beta) = & 14.9229 + 2.7866\alpha + 0.0137\beta + 1.6389\gamma \\ & - 1.2883\eta\delta - 0.7379\eta\epsilon - 0.0092\delta\epsilon \\ & - 0.4379\alpha^2 + 0.7649\alpha\beta + 1.0771\beta^2\end{aligned}\quad (4.8)$$

The standard errors of the parameter estimates in (4.8), and the associated P values of the corresponding test statistics are given in Table 4.5. We note that all the terms in the model are highly significant, with the exception of the coefficient of β .

Using model (4.8) with $\mathcal{F}(x_0^T\beta)$ given as in (4.8), we can compute the estimated probabilities of a success outcome of y_1^T at the 17 factorial combinations of $\{N, \beta, \rho\}$ in Table 4.2. These probabilities are given in Table 4.6. Since there are several replicates of $y = \mathcal{F}(x_0^T\beta) < 0$ at each factorial combination, the maximum difference in

Table 4.6 The estimated $P(\chi^2_{\alpha} < \hat{\chi}^2, g|g)$ using equation (4.4) and its maximum and average differences with $\hat{p} = P(\chi^2_{\alpha} < \hat{\chi}^2)$ values for a given value of $g = \{0.1, 0.5, 1\}$

N	$n = \log_2(N)$	α	$\hat{\alpha}$	$\hat{\chi}^2(\alpha)$	$\max\{g(\alpha) - \hat{p}\}$	average diff
15	3.224	0.05	0.0	0.0784	0.0708	0.0076
			0.4	0.0540	-0.0330	0.0002
			0.8	0.0307	0.0413	0.0099
	0.10	0.0	0.0	0.0707	0.0614	0.0093
			0.4	0.0463	-0.0375	-0.0075
			0.8	0.0219	0.0500	0.0281
	0.40	0.0	0.0	0.0500	0.0440	0.0060
			0.4	0.0250	-0.0100	-0.0050
			0.8	0.0050	0.0000	0.0000
30	3.912	0.05	0.0	0.2042	0.1848	0.0194
			0.4	0.1427	-0.0393	0.0004
			0.8	0.0717	0.0003	0.0000
	0.10	0.0	0.0	0.1811	0.0990	0.0807
			0.4	0.1209	-0.0330	-0.0008
			0.8	0.0500	0.0000	0.0000
	0.40	0.0	0.0	0.0812	0.0000	0.0000
			0.4	0.0340	-0.0000	-0.0000
			0.8	0.0050	0.0000	0.0000
75	4.237	0.05	0.0	0.3080	0.1807	0.0000
			0.4	0.0710	-0.0070	0.0000
			0.8	0.0010	0.0000	0.0000
	0.10	0.0	0.0	0.2840	0.0900	0.0000
			0.4	0.0300	-0.0070	0.0000
			0.8	0.0000	0.0000	0.0000
	0.40	0.0	0.0	0.0700	0.0000	0.0000
			0.4	0.0000	0.0000	0.0000
			0.8	0.0000	0.0000	0.0000

will be the average difference, between the value of $\hat{p}$ in Table 4.2 and the corresponding estimated probability at each (N, k, p) is computed to check the adequacy of fit. The results are also shown in Table 4.3. We note that values of the maximum difference range from -0.0033 to 0.0034, and the values of the average difference range from -0.0004 to 0.0001. These values along with those of the scaled deviance and scaled chi-squared statistics in Table 4.3 indicate that model (4.8), with $f^*(\mathbf{x}|\beta)$ given by (4.6), provides a good fit to the data in Table 4.2. It should be noted that using the canonical link function for the gamma distribution, namely the inverse function, instead of the logarithmic link, produced unsatisfactory results. This is caused by negative values assumed by $f^*(\mathbf{x}|\beta)$ at some of the above-estimated factorial combinations in Table 4.2, which is not possible since with an inverse link, $f^*(\mathbf{x}|\beta)$ would be the reciprocal of $10^p[g(\mathbf{x})/1 - g(\mathbf{x})]^2$.

The modeling of the probability of a negative estimate of σ_0^2 using (4.6) in combination with (4.8) provides a better insight into the effects of N , k , and p on the quality of APQVA estimators of σ_0^2 . Contour plots of $g(\mathbf{x})$ in (4.6) can be used for this purpose. These plots are shown in Figures 4.1, 4.2, and 4.3. We can clearly see that the probability of a negative estimate decreases with increasing values of r , k , and p (recall that $r = \log(N)$). In particular, for fixed values of r and k , as in Figure 4.1, the probability increases as the data set becomes more unbalanced, that is, as k decreases. Note, however, that a k value as low as 0.05 can still result in relatively small negative probability values.

4.1.4. The use of model (4.8) for design comparison

Irwin, Gaudin, and McCulloch (1982, pp. 70-76) considered two unbalanced designs for the one-way random model with $N = 25$ and $k = 5$. The purpose was to study the effect of unbalance on the variance of the APQVA estimator of σ_0^2 . The

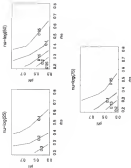


Figure 4.1. Contour plots of the estimated $P(x_i^* < 0)$ for each level of $v = \log(N)$

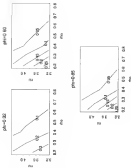


Figure 4.2: Contour plots of the estimated $P(r_0^* < 0)$ for each level of $p = -\log(N)$

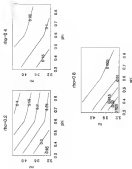


Figure 4.3: Contour plots of the estimated $P(x_0^* < 0)$ for each level of $\rho = \log(N)$

two designs are D_{42} and D_{52} , which were defined in (3.20) and (3.21), respectively. The corresponding values of ϕ are $\phi_{42} = 0.51$ for D_{42} and $\phi_{52} = 0.26$ for D_{52} . Thus D_{42} is more imbalanced than D_{52} . Along with these designs let us also consider the balanced design, D_6 in (3.22) for which $\phi_6 = 1$. For given values of $\eta = \frac{1}{10}$, and hence of $\rho = \frac{1}{10}$, we can calculate the exact value of $P(\mu_1^2 < 0)$ for each of the above mentioned three designs using Davies' [1985] algorithm. The corresponding minimums of $P(\mu_1^2 < 0)$ are obtained by using model (4.1) along with model (4.3). The results are given in Table 4.7. Note that for $\phi_6 = 1$, $\phi_{42} = 0.51$, and some of the ρ values in this table, the use of model (4.3) amounts to an interpolation since such values fall outside the region C in (4.3). Nevertheless, the exact values of $P(\mu_1^2 < 0)$ are quite close to their corresponding predicted values, as this can be seen from Table 4.7. Thus the prediction capability of model (4.4) is quite good, even at points outside the region C . From Table 4.7 we note that the probability of a negative estimate is smaller with D_{42} than with D_{52} . The values for D_6 are smaller than those for D_{42} and D_{52} , as expected.

4.1.5. Concluding remarks

The use of generalized linear models, in combination with a method to generate designs having a specified degree of imbalance, can be quite effective in studying the behavior of $P(\mu_1^2 < 0)$ for the one-way random model. Exact values of such a probability can be obtained using Davies' [1985] algorithm. These values are used to fit a model in which the response is $P(\mu_1^2 < 0)$ and the control variables are n , ρ and ϕ , where $n = \log_2(N)$, N is the total number of observations, ϕ is the measure of imbalance, and $\rho = \frac{1}{\sqrt{1+\eta}}$. If such a model is deemed adequate, then it can be used to determine the effect of changing N , ϕ and ρ on $P(\mu_1^2 < 0)$, and hence on the quality of ANOVA estimations.

Table A.7: Prediction capability of model (A.6) for designs D_5 , D_{10} , and D_{20}

$$D_5 = \{5, 5, 5, 5, 5\} \quad N = 20 \quad p = 3 \quad P_0 = 3 \quad \phi_0 = 1 \quad \phi_1 = \frac{1}{10} \quad p = \frac{1}{10}$$

$\eta\left(\frac{1}{10}\right)$	ρ	$\hat{\rho}(\rho)$	$P(\hat{\rho}_0^2 < 0)$	RMSE
10	0.9979	0.999370	0.00000	0.00196
5	0.9997	0.99970	0.00000	0.00046
1	0.9999	0.999940	0.00000	0.00017
0.5	0.9999	0.99999	0.00000	0.00006
0.25	0.9999	0.99999	0.00000	0.00003
0.1	0.9999	0.99999	0.00000	0.00002

$$D_{10} = \{1, 1, 1, 1, 1\} \quad N = 20 \quad p = 3 \quad P_0 = 3 \quad \phi_0 = 3 \quad \phi_1 = \frac{1}{10} \quad p = \frac{1}{10}$$

$\eta\left(\frac{1}{10}\right)$	ρ	$\hat{\rho}(\rho)$	$P(\hat{\rho}_0^2 < 0)$	RMSE
10	0.9999	0.99970	0.00000	0.00174
5	0.9997	0.99950	0.00000	0.00066
1	0.9999	0.99994	0.00000	0.00019
0.5	0.9999	0.99999	0.00000	0.00008
0.25	0.9999	0.99999	0.00000	0.00004
0.1	0.9999	0.99999	0.00000	0.00002

$$D_{20} = \{1, 1, 1, 1, 1\} \quad N = 20 \quad p = 3 \quad P_0 = 3 \quad \phi_0 = 3 \quad \phi_1 = \frac{1}{10} \quad p = \frac{1}{10}$$

$\eta\left(\frac{1}{10}\right)$	ρ	$\hat{\rho}(\rho)$	$P(\hat{\rho}_0^2 < 0)$	RMSE
10	0.9999	0.99997	0.00000	0.00006
5	0.9997	0.99994	0.00000	0.00004
1	0.9999	0.99997	0.00000	0.00001
0.5	0.9999	0.99997	0.00000	0.00001
0.25	0.9999	0.99997	0.00000	0.00001
0.1	0.9999	0.99997	0.00000	0.00001

$\hat{\rho}(\rho)$ is the predicted value of $P(\hat{\rho}_0^2 < 0)$

It is also possible to extend the proposed methodology to higher-order models. Elston (1987) introduced a general technique to measure imbalance for any nested or crossed-classification models. The extension, however, requires generating designs with specified degrees of imbalance for each model. The method introduced by Elston (1984) applies only to the two-way model. Extending this method to crossed-classification models is straightforward. However, for nested models, or models with crossed and nested effects, the extension can be more involved.

Methods of estimation of variance components other than that of ANOVA can also be addressed with this proposed methodology. For example, we can consider maximum-likelihood variance component estimation (MLVCE), which may produce any norm estimator (see Seneta, Casella, and McCulloch 1984, Ch. 11, Rao and Kishin, 1964, Ch. 4). In this case, David's algorithm cannot be used. Instead, probability of negative estimation of variance components may be obtained by simulation.

4.2 The Power of an *F*-test With Heterogeneous Error Variance

4.2.1 Introduction

The statistical tests of variance components for a balanced model have nice properties. More specifically, the sums of squares in the ANOVA table are independently distributed as scaled chi squared variates under the assumptions of normality, homogeneity of variances, and independence of the model/interaction effects. This enables us to have exact tests for variance components. We can also have optimum tests like uniformly most powerful (UMP), uniformly most powerful unbiased (UMPU), uniformly most powerful invariant (UMPI), and uniformly most powerful invariant unbiased (UMPIU). These tests always coincide with the ANOVA-based exact tests whenever such tests exist. When no exact test exists for testing a variance component, we can

always consistent as approximate F -test using Scheffé-Sidak's procedure (Scheffé-Sidak, 1941-1942). Kiefer's (1955a, 1955b) work concerning the approximation is cited. We also would like to mention Sedner's tests, which often yield exact unbiased tests when exact optimum tests do not exist (Sedner, 1964).

In the case of an unbalanced model, exact tests of variance components are available for a limited number of models. These exact F -tests are based on an idea originally due to Wald (1947) and are extended by Searl and El-Saoudy (1964). However, the existence of a uniformly-optimum test is not for the unbalanced model, and hence locally-optimum tests have been developed. For example, in the case of an unbalanced one-way random model, there is no uniformly optimum test for testing the variance component of a treatment factor, but a locally best invariant (LBI) test is available (Das and Senka, 1967). On the other hand, El-Saoudy and Searl (1962) proposed a modified harmonic mean test, and showed by comparing empirical powers that it outperforms the locally most powerful (LMP) test and Wald's F -test when the design is extremely unbalanced. Therefore a study of the power of a test is still of interest. For more details concerning exact and optimum tests, we refer to Ferguson (1967), Lehmann (1966), Kiefer and Senka (1963), and Kiefer, Wolfowitz and Senka (1968).

Note that the power of the ANOVA-based F -test depends on the design used and the values of variance components. Dwyer and Kaud (1961) studied the effect of unbalance on the power of the F -test for the one-way random model. They compared the empirical power to the corresponding likelihood ratio test (LRT), using designs having various degrees of unbalance. They showed that the LRT is more powerful when the design is extremely unbalanced. In Section 4.2, we show that the exact value of the power of the F -test can be obtained by using Dwyer's (1961) algorithm since the power can be expressed as a probability of linear combinations of independent chi-squared variates. Using this exact value, we model the power in terms of the design

and the variance components to get a better insight of the effect of these factors on the power. By this, we can investigate the effect of imbalance on the power of the F -test for a variance component. Moreover, the proposed modeling technique can be used effectively when an experimenter wants to find a design which ensures a certain level of the power of an F -test. This can provide a guideline to an experimenter to design statistical experiments. In Section 4.3.3, the power of the F -test for a variance component with one-way random model is formulated. This power value is modeled using generalized linear model technique in Section 4.3.4. To perform this analysis, we employed the same generated designs which were used in Section 4.1. Concluding remarks are given in Section 4.3.4.

4.3.3 Power of the F -test for σ_1^2 for the one-way model

Consider the unbalanced one-way random model (3.1) and its vector form as (3.3). The variance-covariance matrix of $\mathbf{y}$, the vector of observed values, which is denoted by Σ , is shown in (3.4). The sum of squares for the treatment factor and the error term are $\mathbf{y}'\mathbf{Q}\mathbf{y}$ and $\mathbf{y}'\mathbf{R}\mathbf{y}$, where $\mathbf{Q}$ and $\mathbf{R}$ are defined as (3.5) and (3.6), respectively. These two sums of squares are independent and $\frac{1}{b}\mathbf{y}'\mathbf{R}\mathbf{y}$ has a central χ^2 -distribution with $R = b$ degree of freedom. Also, under $H_0: \sigma_1^2 = 0$, $\frac{1}{b}\mathbf{y}'\mathbf{Q}\mathbf{y}$ follows a central χ^2 -distribution with $b - 1$ degree of freedom. This gives us a test statistic for testing the hypothesis $H_0: \sigma_1^2 = 0$ as

$$F = \left(\frac{R - b}{b - 1} \right) \frac{\mathbf{y}'\mathbf{Q}\mathbf{y}}{\mathbf{y}'\mathbf{R}\mathbf{y}}, \quad (4.10)$$

which, under H_0 , has the F -distribution with $b - 1$ and $R - b$ degrees of freedom. This test is UMPU if the data are in balanced.

As was mentioned in Section 3.2, the treatment sum of squares, $\mathbf{y}'\mathbf{Q}\mathbf{y}$, can be expressed as a linear combination of independent central chi-squared variables of the

form

$$x'Qx = \sum_{i=1}^r \lambda_i x_{i0}^2, \quad (4.11)$$

where $\lambda_1, \lambda_2, \dots, \lambda_r$ are the distinct positive eigenvalues of QE with multiplicities $m_1, m_2, \dots, m_r$, respectively. Using the relationship in (4.11), the power of the F -test for the hypothesis $H_0: \alpha = 0$ can be expressed as

$$\begin{aligned} F &= P(F > F_{1-\alpha, r, n-r} \mid \sigma_0^2 \neq 0) \\ &= P\left[\sum_{i=1}^r \lambda_i x_{i0}^2 > \left(\frac{r-1}{n-r}\right) F_{1-\alpha, r, n-r} \sigma_0^2 \mid \sigma_0^2 \neq 0\right] \\ &= P\left[\sum_{i=1}^r \lambda_i^2 x_{i0}^2 - \lambda_{i-1}^2 \left(\frac{1-\rho^2}{\rho}\right) x_{i-1,0}^2 > 0 \mid \sigma_0^2 \neq 0\right], \end{aligned} \quad (4.12)$$

where $\lambda_i^2 = \frac{1}{\lambda_i}$, $i = 1, 2, \dots, r$, which are the distinct positive eigenvalues of QE^2 with multiplicities m_i , $i = 1, 2, \dots, r$, respectively, such that $\sum_{i=1}^r m_i = k-1$. The matrix $E^2 = \frac{1}{2}E$ is shown in (4.1). Also $\lambda_{i-1}^2 = \frac{1}{\lambda_{i-1}} F_{1-\alpha, r, n-r}$ and $\rho = \frac{1}{\sqrt{2(n-2)}}$. The exact value of (4.12) can be obtained, again by using a computer algorithm given by Davies (1986). This exact value can be modeled in terms of N , k and ρ , in order to explain a deeper insight into the effect of these variables on the power.

4.5.3. The Effect of Imbalance On the Power of the F -test

A design, that is a set of n_i values for the model (3.1), can be generated for given values of N , k , and δ using Kiefer's (1986) method, as was stated in Section 4.1.5. With the value of $\rho = \frac{1}{\sqrt{2(n-2)}}$ and the generated design Ω , we can compute an exact value of the power of the F -test in (4.12). Since this power is not an explicit function of the parameters N , k and ρ for a given $k-k$, will be modeled as a response function of these parameters.

Let us, for example, consider $k = 2$ so that $D = \{n_1, n_2, n_3, n_4, n_5\}$. Consider a 2^2 factorial arrangement having the selected levels of $N = 25, 50, 75$, $\beta = 0.50, 0.60, 0.80$ and $\rho = 0.1, 0.4, 0.6$. The 303-design generated in Section 4.1, which are listed in Table 4.1, are used in this study. The power in (4.12) is calculated using Boxer's (1984) algorithm for each of the generated designs. In this study, $\alpha = 0.05$ is used. Part of the generated designs and the corresponding powers are listed in Table 4.3.

Since several designs are generated for each specification of N and β , we have replicates of the power value for each combination of N , β and ρ . The data in Table 4.3 can now be used to model the power, g , as terms of these three variables, N , β and ρ . Considering the nature of the response g , being a probability, it would be appropriate to consider a generalized linear model fitting, as it is stated in Section 3.1.4. Since $0 < \beta \leq 1$ and $0 < \rho < 1$, we consider the logarithmic transformation of N $x = \log(N)$, instead of N itself. The range of the variable x is about 1.1, which is comparable to the ranges of β and ρ . The region of experimentation, C , is described in (4.3). Let us now consider the transformation

$$w_m = \left[\frac{1-\beta}{\beta} \right]^m \quad m \geq 1$$

which maps the interval $(0, 1]$ onto $[0, \infty)$. Thus we have replicates of g for each combination of inputs (x, β, ρ) , the sample mean $\bar{w}_m$ and the sample variance $s_{w_m}^2$ of w_m for the i^{th} input $i = 1, 2, \dots, 18$, can be computed for various values of $m = 1$. Simple linear regression model without an intercept using $\bar{w}_m$ and $s_{w_m}^2$ as $\bar{w}_m$ and $s_{w_m}^2$ is fitted to present a relationship between the mean and standard deviation of w_m or between the mean and the variance of w_m , respectively. The coefficient of determination, R^2 using $\bar{w}_m$ and $s_{w_m}^2$ is 0.93 for $m = 1$, and 0.91 for $m = 2$, whereas that of using $\bar{w}_m$ and $s_{w_m}^2$ is 0.75 for $m = 1$ and 0.59 for $m = 2$. This indicates that there is a strong linear relationship between the mean and standard deviation of w_m

with $m = 2$. Therefore, with the transformation,

$$w = \left[\frac{1 - \beta}{\beta} \right]^{\frac{1}{\beta}}. \quad (4.12)$$

The gamma distribution can be chosen as a probability distribution for the random variable w . Let us denote the mean and variance of w by μ_w and σ_w^2 , respectively. We choose $\xi = f'(x)/\beta$ as the linear predictor, where β is a vector of unknown parameters, and $f(\mathbf{x})$ is a scalar function of $\mathbf{x}$ such that $f'(\mathbf{x})/\beta$ is a polynomial in $\mathbf{x} = (x_1, x_2)$ of a certain degree. To relate the linear predictor ξ to the mean response μ_w , we consider the logarithmic link function for random variables as those stated in Section 3.1.4. We thus have the following components for our model fitting:

- (a) For the random component, the response variable w in (4.12) has independent gamma distribution with a mean μ_w and a variance function $V(\mu) = \mu_w^2$.
- (b) For the systematic component, the linear predictor ξ is given by $\xi = f'(\mathbf{x})/\beta$.
- (c) The link function which relates μ_w to ξ is the logarithmic function, $\xi = \log(\mu_w)$.

For the linear predictor, ξ , we consider the second degree model:

$$f'(\mathbf{x})/\beta = \beta_0 + \beta_1 x + \mathbf{x}' \mathbf{B} \mathbf{x}$$

where $\beta_1' = (\beta_1, \beta_2, \beta_3)$ and $\mathbf{B} = \{\beta_{ij}\}$, $i, j = 1, 2, 3$. The GENMOD procedure in SAS (1987) is used to fit various model forms. The scaled deviance and the scaled Pearson's chi-squared statistic of the fitted model can be used as measures of the goodness of fit. Based on these values and the significance of Wald's test, statistics for each parameter, the model selected to represent the linear predictor ξ is

$$\begin{aligned} f'(\mathbf{x})/\beta &= 11.3432 - 1.2613x - 3.2718y - 3.4253y^2 \\ &\quad + 1.6228xy + 1.6228xy^2 - 1.4228y^3 \\ &\quad + 2.4753y^2 - 1.8753y^3 \end{aligned} \quad (4.14)$$

Table 4.5 The standard errors of parameter estimation and goodness-of-fit statistics for model (4.14)

Fixed: $\beta_{\text{mean}} = 394.7975$, $\beta_0 = 394$ (value of $\beta = 1.0307$)						
Fixed: χ^2 value = 228.0288, $\chi^2_{\text{df}} = 394$ (value of $\chi^2 = 1.0770$)						
variable	parameter	estimate	est. var.	χ^2 value	P-value	
μ	β_0	11.8385	8.4675	149.4575	0.0004	
	β_1	3.8895	8.1020	158.7938	0.0004	
	β_2	3.2718	8.7958	158.2731	0.0004	
	β_3	3.4982	8.6759	148.6988	0.0004	
σ^2	β_{SD}	3.8880	8.1068	148.4983	0.0004	
σ^2	β_{SD}	3.8884	8.1054	175.7959	0.0004	
σ^2	β_{SD}	4.4555	8.2584	168.3948	0.0004	
σ^2	β_{SD}	5.4758	8.2680	158.2638	0.0004	
σ^2	β_{SD}	1.8728	8.2887	22.7333	0.0004	

The standard errors and the associated P-values of the corresponding test statistics are shown in Table 4.5. Note that all terms are highly significant.

Let $\hat{g}(x)$ denote the estimated value of g in (4.13) as x given μ inside the region of experimentation $\mathcal{C}$ in (4.7). Using model (4.14), it can be calculated, at a given design point μ , by

$$\hat{g}(x) = \frac{1}{1 + e^{(\beta'x)/\sigma^2}} \quad (4.15)$$

These estimated values of g for given specifications of μ are listed in Table 4.6. Since there are several replications of g at each factorial combination, the maximum difference, as well as the average difference, between the power values of g in Table 4.2 and the corresponding estimated power, $\hat{g}(x)$, at each input (x, μ, σ^2) is calculated. The results are shown in Table 4.10. We note that values of the maximum difference range from 0.0001 to 0.0005 and values of the average difference range from 0.0027 to 0.0001. These values are very small as compared to the size of the actual power.

Table 4.20: The estimated power $\hat{p}(\alpha)$, of the F test using model (4.13) and its confidence and average difference with power, p , for given values of $\alpha = \{0.01, 0.1\}$

N	$\alpha = \log(N)$	α	ρ	$\hat{p}(\alpha)$	$\text{rate}[\hat{p}(\alpha) - p]$	average diff
25	3.201	0.20	0.0	0.17790	0.01190	0.00475
			0.4	0.37943	0.00000	0.00437
			0.6	0.54694	-0.00000	0.01374
		0.40	0.2	0.20732	0.00000	0.00333
			0.4	0.52004	0.01373	0.00000
			0.6	0.70637	-0.00000	0.01000
	0.40	0.2	0.20687	0.01000	-0.00000	
			0.4	0.50409	-0.00000	-0.00000
			0.6	0.70133	0.00000	0.00000
50	3.693	0.20	0.2	0.22170	0.01731	0.00000
			0.4	0.54786	0.00000	0.00000
			0.6	0.64390	-0.00000	-0.00000
		0.40	0.2	0.44174	-0.00000	-0.00000
			0.4	0.70000	0.00000	-0.00000
			0.6	0.74000	0.00000	0.00000
	0.40	0.2	0.24334	0.00000	0.00000	
			0.4	0.70000	0.00000	-0.00000
			0.6	0.70000	0.00000	0.00000
75	3.877	0.20	0.2	0.42340	0.00000	0.00000
			0.4	0.70000	0.00000	0.00000
			0.6	0.60000	0.00000	0.00000
		0.40	0.2	0.61630	0.00000	-0.00000
			0.4	0.60000	-0.00000	-0.00000
			0.6	0.60000	0.00000	-0.00000
	0.40	0.2	0.64330	0.00000	0.00000	
			0.4	0.60000	0.00000	0.00000
			0.6	0.60000	0.00000	0.00000

values. These values and the products of fit test statistics in Table 4.4 indicate model (3.10) with $f^*(x)\beta$ given by (4.14) explains the data in Table 4.1 very well.

To see the effect of the explanatory variables x on the power of the F -test, contour plots of $g(x)$ are displayed in Figures 4.3, 4.5, and 4.6. The power increases as we increase values of x , β , and ρ . However, for a fixed value of one variable, we can increase the power by increasing a value of another variable and decreasing a value of the other variable. For example, we see in Figure 4.3(a) that, when $x = \log(32)$, the estimated power is 0.3 when $\rho = 0.3$ and $\beta = 0.45$, and it reaches 0.8 when $\rho = 0.7$ and $\beta = 0.45$. Thus, when $N = 10$ (i.e., $x = \log(32)$), we can increase the power from 0.3 to 0.8 by increasing ρ value from 0.3 to 0.7 but decreasing β value from 0.45 to 0.45. For more practical reports, we can have the following observations by examining Figure 4.3(a) and 4.5(a): when $\rho = 0.4$, in order to get power value of 0.8, we need a design with $N = 75$ (i.e., $x = 4.3$) observations whose means of imbalances is almost as high as $\beta = 0.8$ [see Figure 4.5(a)]. However, when $\rho = 0.7$ a design with $N = 30$ (i.e., $x = 3.4$) and $\beta = 0.4$ is sufficient to get the power value 0.8 [see Figure 4.5(a)]. For another report, if we like to ensure the power of an F -test is to be at least 0.8, a design having a degree of imbalance as low as $\beta = 0.4$ is good enough when we have $\rho = 0.8$ and $N = 30$ (i.e., $x = 3.4$) [see Figure 4.5(a)]. These observations can provide very practical guidelines to an experimenter who is in a stage of designing a statistical experiment.

Now, the question is how good the model is to predict the power of the F -test at points outside the experimental region C in (4.7). To answer this question, we again consider three designs, D_1 , D_2 , and D_{25} , which were used in Section 4.1.3 for similar purposes to investigate the effect of imbalance on the probability of a negative ANOVA estimator. For each design, the exact power of the F -test for given values of $\eta = \frac{\beta^2}{\sigma^2}$ (i.e., $\rho = \sqrt{\frac{\beta^2}{\sigma^2 + \beta^2}} = \sqrt{\frac{\eta}{1 + \eta}}$) is calculated using Devore (2008) algorithm. Also, the corresponding estimated power $\hat{g}(x)$ based on the model (3.15) with $f^*(x)\beta$

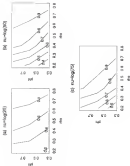


Figure 4.4: Contour plots of the estimated power of the F-test for no among-group variance components for each level of $\alpha = \log(N)$.

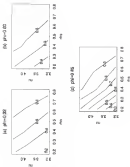


Figure 4.3—Contour plots of the estimated power of the F test for no among-group variance component for each level of $\alpha + \ln(\log N_0)$

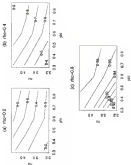


Figure 4.8: Contour plots of the estimated power of the F-test for no among-group variation (empirical) for each level of ρ [$\rho = \log(N)/N$].

given by (4.14) is calculated. These values along with their differences are listed in Table 4.11. Note that $\phi_1 = 1$ for the design D_1 , $\phi_{12} = 0.20$ for the design D_{12} and some p values ($p = 0.0001$, $p = 0.0002$) fall outside the region C in (4.7). Nevertheless, the predicted powers with model (4.10) are very close to their corresponding exact values. Thus the predictive capability of model (4.11) along with model (4.14) is good, even at points outside the region C . As we know, for a balanced design, the F-test is UMP and therefore the powers for D_1 should be larger than those for D_{12} and D_{21} . We can see this from the Table 4.11. We also note that the power is larger with D_1 than with D_{12} for all of p values.

6.2.4. Concluding remarks

With an unbalanced one-way random model, the ANOVA-based F-test for testing the significance of the random effect variance-component is valid. However, this test does not have any optimum properties. There is also a locally optimum test which is different from the F-test. However, the use of the F-test is still popular due to its simplicity. In that respect, the power of the F-test is always of interest for an experimenter.

The power of the F-test for a variance component with an one-way random model depends not only on the design used, but also on its variance-components. The exact value of the power can be obtained using, for example, Dennis' (1988) algorithm. In Section 4.3, the power is modeled as a function of the total number of observations, the degree of unbalance, and the ratio of the variance components. The fitted model can be used to see the effect of these variables on the power, and hence on the ANOVA-based F-test. Moreover, the fitted model in (4.10) along with model (4.14) can provide guidelines for choosing a design capable of producing desirable power values. This modeling technique can therefore be used effectively at the designing stage of a statistical experiment.

Table 4.11: Prediction capability of model [4.11] for design D_{10} , D_{100} and D_{101}

$D_{10} = \{1, 1, 1, 1, 1\}$, $N = 10$, $p = 0.20$, $d_0 = 1.0$, $p = \frac{1}{10}$

$\alpha \left(= \frac{p}{1-p} \right)$	p	$g^*(\alpha)$	p	$g^*(\alpha) - p$
0	0.0000	0.0000	0.0000	-0.0000
1	0.0007	0.0010	0.0000	0.0010
1	0.0008	0.0009	0.0000	-0.0009
0.5	0.3333	0.0700	0.0700	-0.0000
0.50	0.3300	0.0690	0.0690	-0.0000
0.5	0.0000	0.0000	0.0000	0.0000

$D_{100} = \{1, 1, 1, 1, 1\}$, $N = 10$, $p = 0.20$, $d_0 = 0.100$, $p = \frac{1}{10}$

$\alpha \left(= \frac{p}{1-p} \right)$	p	$g^*(\alpha)$	p	$g^*(\alpha) - p$
0	0.0000	0.0000	0.0000	-0.0000
1	0.0007	0.0007	0.0000	0.0007
1	0.0008	0.0000	0.0000	0.0000
0.5	0.3333	0.0007	0.0007	0.0000
0.50	0.3300	0.0007	0.0007	0.0000
0.5	0.0000	0.0000	0.0000	0.0000

$D_{101} = \{1, 1, 1, 1, 1\}$, $N = 10$, $p = 0.20$, $d_0 = 0.000$, $p = \frac{1}{10}$

$\alpha \left(= \frac{p}{1-p} \right)$	p	$g^*(\alpha)$	p	$g^*(\alpha) - p$
0	0.0000	0.0000	0.0000	-0.0000
1	0.0007	0.0000	0.0000	0.0000
1	0.0008	0.0000	0.0000	0.0000
0.5	0.3333	0.0007	0.0000	0.0007
0.50	0.3300	0.0000	0.0000	0.0000
0.5	0.0000	0.0000	0.0000	0.0000

* $g^*(\alpha)$ is the predicted value of the power, p

The extension of the proposed methodology to higher-order models is of interest. Investigation of the effect of imbalance on the power of tests other than the F test is also of interest. For example, the performance of the modified likelihood ratio test suggested by El-Daouy and Isely (1993) can be examined more thoroughly in terms of the power using the methodology shown in this review. In this case, however, we cannot use the Davies algorithm to calculate the power of the test, but it can be obtained by computer simulation.

4.4. The Power of an Exact Wald Homogeneous Error Variance

4.4.1. Introduction

In many experimental situations, the error variances in a given random model may be homogeneous due to a variety of reasons. Inequality of such variances can seriously affect an ANOVA test, particularly when the design under consideration is unbalanced (see, for example, Miller 1966, Section 4.7). This problem has been investigated by many authors for fixed-effects models (see, for example, Brown and Perryman 1934, Bishop and Duckworth 1978, Levy 1976a, 1976b, Scovakoff, 1944, Silverman 1965). Fewer articles, however, considered situations involving random or mixed models. Rao, Kishor, and Cochran (1961) discussed point estimation of the parameters of the one-way random model with unequal error variances. Agresti and Gibbons (1983) provided an approximate test for the hypothesis $H_0: \sigma_1^2 = 0$ using the Studentized variate approximation to the distribution of the among-group sum of squares for the one-way random model. Singh and Isely (1993) investigated the effects of heterogeneity and data imbalance on the probability of a correct estimate of σ_1^2 . Singh (1993) studied the same effects with regard to the power of the usual F test concerning the hypothesis $H_0: \sigma_1^2 = 0$. Power values were computed

using a doubly infinite series of incomplete beta functions. We concluded that heteroscedasticity increases the power of the test as long as where the larger variances are associated with the smaller group sizes. If the larger variances are associated with the larger group sizes, the power is decreased. A small increase in the power, due to heteroscedasticity, was noted with balanced data.

In Section 4.3, we model the power of the usual F -test concerning σ_1^2 especially in terms of σ_1^2 and the error variances, as well as certain design parameters. The modeling scheme provides added insight into the effects of unbalance and heterogeneity of error variances on the power of the F test. In particular, the empirical model is conventionally used to determine conditions resulting in maximum or minimum power values. Furthermore, using plots of the so-called power traces, it is possible to detect influential error variances and assess their impact on the power. The power of the usual F test for a variance comparison of the one-way random model with unequal error variances is formulated in Section 4.3.1. In Section 4.3.2, the power value is modeled using generalized linear model techniques. To do this a number of designs are presented using Box-Cox's (1958) method. Chemical analysis is performed in Section 4.3.4 to determine the nature of the stationary point and the entire response surface. Optimum values of the predicted power is obtained in Section 4.3.5. Section 4.3.6 is devoted to introducing the power trace plot. Finally, modeling results are given in Section 4.3.7.

4.3.2. Comparison of power

Consider the one-way random model

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad (4.3.1)$$

$i = 1, 2, \dots, k$, $j = 1, 2, \dots, n_i$, where α_i and ϵ_{ij} are independently distributed as normal variates with zero means and variances, $\sigma_1^2 = \sigma_1^2$ ($i = 1, 2, \dots, k$), respectively. We note that the error variances are equal within groups, that is, for fixed i , but are

unequal for different values of i . Model (3.13) is written as

$$y = \mu\mathbf{1}_N + [\sigma_{i_1}^2 \mathbf{1}_{n_1}] \otimes \mathbf{1} + \varepsilon \quad (3.17)$$

where $\mathbf{1} = \sum_{i=1}^k n_i \mathbf{1}_{n_i}$ is a column vector of n_i ones ($i = 1, 2, \dots, k$), $\mu = [\alpha, \alpha_{n_1}, \dots, \alpha_k]'$, and y and ε are the vectors of observations and random errors, respectively. The variance-covariance matrix of y , denoted by Σ , is of the form

$$\Sigma = \sigma_0^2 \mathbf{I}_{N+1}^* + \sigma(\mathbf{1}_N \mathbf{1}_N') \quad (3.18)$$

The usual F -test for testing the hypothesis $H_0: \sigma_0^2 = 0$ is

$$F = \left(\frac{N-k}{k-1} \right) \frac{\mathbf{1}'\mathbf{Q}\mathbf{y}}{\mathbf{y}'\mathbf{R}\mathbf{y}} \quad (3.19)$$

where $\mathbf{Q}$ and $\mathbf{R}$ are defined as (3.7) and (3.8), respectively. This test is based on the assumption that the error variances, that is, the σ_i^2 's, are equal. In this case, F has under H_0 the F -distribution with $k-1$ and $N-k$ degrees of freedom.

Let us consider the power of the F -test in (4.19) when in reality the error variances are not necessarily equal. In this case, the variance-covariance matrix Σ has the form given by formula (4.18). The power function is

$$P(F \geq P_{\alpha, k-1, N-k} | \sigma_0^2 \neq 0) = P(\mathbf{y}'\mathbf{Q}\mathbf{y} \geq h_\alpha \mathbf{R}\mathbf{y} | \sigma_0^2 \neq 0), \quad (4.20)$$

where $h_\alpha = \frac{1}{2} [F_{\alpha, k-1, N-k}]$ and $P_{\alpha, k-1, N-k}$ is the upper α -percentile of the F -distribution with $k-1$ and $N-k$ degrees of freedom. Since y is normally distributed, the quadratic form $\mathbf{y}'\mathbf{Q}\mathbf{y} - h_\alpha \mathbf{R}\mathbf{y}$ can be expressed as

$$\mathbf{y}'\mathbf{Q}\mathbf{y} - h_\alpha \mathbf{R}\mathbf{y} = \sum_{i=1}^r h_i z_i^2 \quad (4.21)$$

where $h_1, h_2, \dots, h_r$ are the distinct nonzero eigenvalues of $(\mathbf{Q} - h_\alpha \mathbf{R})\mathbf{E}_0^{-1}$ with multiplicity $\nu_1, \nu_2, \dots, \nu_r$, respectively, such that $\sum_{i=1}^r \nu_i = \text{rank}(\mathbf{Q} - h_\alpha \mathbf{R})$, and the z_i^2 's are independent central chi-squared variates with ν_i degrees of freedom (see

Johnson and Kotz (1970), p. 155. Note that

$$\begin{aligned}\frac{(\mathbf{Q} - \mathbf{b}_0 \mathbf{M})\mathbf{E}}{s_0^2} &= (\mathbf{Q} - \mathbf{b}_0 \mathbf{M})\mathbf{Z}_{\mathbf{b}_0}^T \left[\mathbf{F}_{\mathbf{b}_0} + \frac{1 - \beta}{\beta} \mathbf{F}_{\mathbf{b}_0} \right] \\ &= \text{trace } \mathbf{F}_{\mathbf{b}_0} = \sum_{j=1}^k \frac{\lambda_j}{\beta + 1}, \quad \beta = 1, 2, \dots, k.\end{aligned}$$

But α gives k the values of λ_j and α_j ($j = 1, 2, \dots, k$) in (4.11) depend on the ANOVA design $D = \{n_1, n_2, \dots, n_k\}$ and on the value β ($\beta = 1, 2, \dots, k$). Formula (4.12) can then be written as

$$P(F \geq F_{1-\alpha, \nu, \beta} | D) = P \left\{ \sum_{j=1}^k \lambda_{\alpha_j} \geq \beta \right\} \quad (4.13)$$

For given k and N , the power value in (4.13) requires the specification of the ANOVA design D and the value β ($\beta = 1, 2, \dots, k$). The right-hand side of (4.13) can be easily evaluated using the computer algorithm by Davies (1986). The degree of imbalance of the design D is determined by a measure of imbalance denoted by ϕ , which was defined in (3.8).

In the next section we show how to develop an empirical relationship between the power, computed using (4.13), and $N, \phi, \nu, \mu_1, \dots, \mu_k$. The purpose of such a relationship is to facilitate the study of the combined effects of imbalance and heterogeneity of the error variances on the power of the F -test in (3.18). A key ingredient in the development of this relationship is a method for the generation of an ANOVA design $D = \{n_1, n_2, \dots, n_k\}$ for specified values of N and ϕ . We refer to N and ϕ as design parameters. A description of this method is given in Kiani (1986). Note that several ANOVA designs can be generated using the same values of N and ϕ , since N and ϕ do not determine the ANOVA design $D = \{n_1, n_2, \dots, n_k\}$ uniquely.

4.3.3 Modeling the point of the Elmer

Suppose that values for K , δ , $p_1, \dots, p_6$ have been specified. Using Khuri's (1986) method, several ANOVA designs, $D = \{n_1, n_2, \dots, n_6\}$, can be generated for the given values of N and k . For each generated D , the values of $n_1, n_2, \dots, n_6$ and the selected values of $p_1, p_2, \dots, p_6$ are used in formula (3.18) to compute the given value which we denote by τ . We can therefore regard τ as a response variable and $N, \delta, p_1, p_2, \dots, p_6$ as its control variables. Thus several ANOVA designs D are generated for each choice of N and k , a number of replications on τ are available at the point $(N, \delta, p_1, p_2, \dots, p_6)$. These replications will be useful in the construction of such a relationship. Given the values of the response τ , it would be appropriate to model τ against $N, \delta, p_1, p_2, \dots, p_6$ using generalized linear models (GLM) techniques.

To demonstrate the development of an empirical model relating τ to N , δ , and $p_1, \dots, p_6$, let us consider as an example that $k=5$. Thus $D = \{n_1, n_2, n_3, n_4, n_5\}$. Let τ denote the point $(N, \delta, p_1, p_2, \dots, p_6)$. For each selection of τ in some region of interest, several determinations on D are made using the corresponding values of N and δ in τ . The values of τ are then computed using Owen's algorithm and formula (3.18). This process results in several replications on τ at the point τ .

Let us now consider the transformation

$$w = \frac{1 - \tau}{\tau}, \quad (3.22)$$

which maps the interval $(0, 1)$ onto $(0, \infty)$. The variable w will be modeled using GLM. For this purpose, we need to specify a distribution for w in addition to a link function g such that $\xi = g(\mu(\tau))$, where $\mu(\tau)$ denotes the mean of the distribution of w at τ , g is a strictly monotone increasing function, and ξ is a linear predictor. Our choice of g is the logarithmic function, $\log(\cdot)$, and that of the distribution of w is the gamma distribution. The choice of this distribution will be justified later. As

for the linear predictor, we consider a second-degree model of the form

$$\xi = \beta_0 + \mathbf{r}'\beta + \mathbf{r}'\beta\mathbf{r}, \quad (4.34)$$

where β_0 and the elements of β and β are unknown parameters. Fitting model (4.34) requires using an appropriate second-degree response surface design. A convenient choice is the Box and Behnken (1960) design for three variables with four center-point replicates. This is an unreplicated 2-level incomplete factorial design. The selected three levels for each of the elements of $\mathbf{r} = [N \ \phi \ \mu_1 \ \mu_2 \ \dots \ \mu_k]$ are: $N = 25, 50, 75$; $\phi = 0.25, 0.50, 0.75$; $\mu_i = 0.1, 0.5, 0.9$, $i = 1, 2, \dots, k$. The second settings (in coded form) of the Box-Behnken design are given in Table 4.12. Note that except for N , the values of all the remaining elements of $\mathbf{r}$ fall inside the unit interval $(0, 1]$. To reduce the range of N , we consider a logarithmic transformation of N : $a_1 = \log(N)$. This way, the range of a_1 for the selected values of N is about 1.1, which is comparable to the ranges of ϕ and μ_i ($i = 1, 2, \dots, k$). Let $a_2 = \phi$, $a_3 = \mu_1$, $a_4 = \mu_2$, $a_5 = \mu_3$, $a_6 = \mu_4$, $a_7 = \mu_5$, and $\mathbf{a} = (a_1, a_2, \dots, a_7)'$. Model (4.34) can then be expressed in terms of $\mathbf{a}$ as

$$\xi = \beta_0 + \mathbf{a}'\beta + \mathbf{a}'\beta\mathbf{a}, \quad (4.35)$$

where β_0 and the elements of β and β are unknown parameters.

For the chosen levels of a_i ($i = 1, 2, \dots, 7$), the region of interest for the fitting of model (4.35) is

$$C = \{\mathbf{a} \mid \log(25) \leq a_1 \leq \log(75), 0.2 \leq a_2 \leq 0.8, 0.1 \leq a_3 \leq 0.9, i = 4, \dots, 7\} \subseteq [0, 1]^7.$$

Using the logarithmic link function in combination with model (4.35), we get

$$\log[\mu(\mathbf{x})] = \beta_0 + \mathbf{a}'\beta + \mathbf{a}'\beta\mathbf{a} \quad (4.37)$$

where $\mu(\mathbf{x})$ is the mean of the distribution of y in (4.35) at the point $\mathbf{x}$. It should be noted here that the logarithmic link was preferred over the reciprocal link, which is the canonical link function for the gamma distribution. The reason for this will be explained later.

Table 4.12 Two-Block designs^a and the number of ANOVA designs generated

M	δ	μ_1	μ_2	μ_3	μ_4	μ_5	Number of designs generated ^{ab}
0	0	0	0.1	0.1	0.1	0	4 for each of 8 design points
0.1	0	0	0	0	0.1	0.1	4
0	0.1	0	0	0.1	0	0.1	4
0.1	0.1	0	0.1	0	0	0	4
0	0	0.1	0.1	0	0	0.1	4 for each of 8 design points
0.1	0	0.1	0	0.1	0	0	4
0	0.1	0.1	0	0	0.1	0	4
0	0	0	0	0	0	0	4 for each of 4 center points

(a) 12 designs for $M = 1$; 8 designs for $M = -1$

(b) 12 designs for $\delta = 1$; 8 designs for $\delta = -1$

(c) 36 designs for $(\mu_1, \mu_2) = (0, 0)$; 34 designs for $(\mu_1, \mu_2) = (1, -1)$; 4 designs for each of $(\mu_1, \mu_2) = (-1, 0), (-1, -1)$

^a The coded settings, $-1, 0, 1$, denote the low, middle, and high levels, respectively, for each variable

^{ab} The total number of ANOVA designs generated is 508

At each point of the Box-Behnken design in Table 4.18, several ANOVA designs are generated using Kuehn's (1966) method. It gives several replications of x , and hence of w , at each point of the Box-Behnken design. The number of generated determinations of $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3, \hat{w}_4, \hat{w}_5)$ corresponding to each point of the Box-Behnken design is also given in Table 4.18. Some of these determinations along with the actual y -value for each generated design are shown in Table 4.19. The corresponding values of r based on the formula (4.22) using Fisher's algorithm, are also given in Table 4.19. Let $\bar{U}_i$ and s_i^2 denote the sample mean and sample variance of the w -replicates at $\mathbf{a}$ corresponding to a point $\mathbf{a}$ in the Box-Behnken design. The task now is to determine if there is a strong linear relationship between $\bar{U}_i$ and s_i^2 or between $\bar{U}_i$ and s_i . Fitting a simple linear regression model with an intercept to the $(\bar{U}_i, s_i^2)$ data results in a coefficient of determination $R^2 = 0.98$. Fitting the same model to the $(\bar{U}_i, s_i)$ data gives $R^2 = 0.97$. This indicates a strong linear relationship between the mean of the distribution of w and its standard deviation, which justifies our choice of the gamma distribution for w .

Fitting model (4.27) to the w -data that were generated at the points of the Box-Behnken design produces a predicted value for w and hence for r (see formula (4.28)). For this case

$$r(\mathbf{x}) = \frac{\hat{w}}{1 + \exp(\beta_0 + \mathbf{x}\beta + \mathbf{x}^T \hat{\mathbf{A}}\mathbf{x})} \quad (4.28)$$

where $r(\mathbf{x})$ denotes the predicted power value at a point $\mathbf{x}$ in the region C defined in (4.26) and $\hat{w}$ and the elements of $\hat{\beta}$ and $\hat{\mathbf{A}}$ are the maximum likelihood estimates corresponding to $\hat{w}$, β and $\mathbf{A}$ respectively. The GENMOD procedure in SAS (1987) was used to fit model (4.27).

Let the i^{th} element of $\hat{\beta}$ and the $(i, j)^{\text{th}}$ element of $\hat{\mathbf{A}}$ in (4.28) be denoted by $\hat{\beta}_i$ and $\hat{A}_{ij}$ respectively ($i = 1, 2, \dots, 7$; $j = 1, 2, \dots, 7$). The tests concerning $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4, \hat{\beta}_5$ and $\hat{A}_{ij}$ were found to be highly insignificant. These terms were therefore dropped

from the model. Model (4.27) was subsequently refitted using the remaining parameters. Estimates of these parameters are given in Table 4.14. We note that all parameters are significantly different from zero. With these parameter estimates, the estimated point values at all points of the Box-Behnken design were computed using model (4.24). The results are shown in Table 4.15. Since there are several replications to y at each design point, the maximum difference, as well as the average difference, between the replicated values and the corresponding estimated value of y (at the same location) is computed to check the adequacy of fit of the model. These values are also given in Table 4.15. It can be noted that the maximum difference ranges from -0.1877 to 0.1258, except at one point where it is equal to -0.2612. The value of average difference ranges from -0.0218 to 0.0044, except at one point where it is equal to -0.0079. These values along with those of the scaled deviance and scaled chi-squared statistic in Table 4.14 indicate that model (4.24) provides a good fit to the data.

It should be noted here that using the canonical link function for the gamma distribution, namely the reciprocal link function instead of the logarithmic link function, produced reliable results. At some points of the Box-Behnken design, the estimated value of y did not fall inside the interval $[0, 1]$.

Note that the range of interest for x_1 (i.e., for y) and $x_2, x = 1, 2, \dots, T$ (i.e., for $A, x = 1, 2, \dots, 5$), in (4.26) cover with range of the parameter space of x 's, namely, $0 \leq x_1 \leq 1, 0 \leq x_i \leq 1, i = 1, 2, \dots, T$. To check the prediction capability of the model (4.26) at points outside the region C especially for the variable x_1 (i.e., for y), we generate designs with selected values of x . These designs along with the values of x are listed in Table 4.16. Note that $N = 15$ and $N = 30$ are the values which fall outside the region C in (4.26). Also, the value of $\phi = 1$, which is for the balanced design case, was not used to build the model (4.26). Hence, the prediction capability of the model (4.26) when $\phi = 1$ is also of interest.

Table 4.12 Some generated GNCAs designs and the corresponding true values of π (π , denotes the actual value of π for a ρ -generated design)

N	ρ	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	π_1	π_2	π_3	π_4	π_5	$\hat{\pi}_n$	π^*
10	0.05	0.5	0.5	0.5	0.5	0.5	10	10	5	5	5	0.0507	0.05000
10	0.05	0.5	0.5	0.5	0.5	0.5	20	10	5	5	5	0.0527	0.05000
10	0.05	0.5	0.5	0.5	0.5	0.5	50	10	5	5	1	0.0494	0.05000
10	0.05	0.5	0.5	0.5	0.5	0.5	10	10	5	5	0	0.0500	0.05000
10	0.05	0.5	0.5	0.5	0.5	0.5	10	10	5	5	2	0.0490	0.05000
10	0.05	0.5	0.5	0.5	0.5	0.5	11	10	10	5	1	0.0510	0.05000
10	0.05	0.5	0.5	0.5	0.5	0.5	10	10	10	5	1	0.0521	0.05000
10	0.05	0.5	0.5	0.5	0.5	0.5	10	11	10	4	2	0.0494	0.05000
75	0.30	0.5	0.5	0.5	0.5	0.5	10	11	5	5	2	0.3008	0.30000
75	0.30	0.5	0.5	0.5	0.5	0.5	10	10	5	5	1	0.3049	0.30000
75	0.30	0.5	0.5	0.5	0.5	0.5	10	11	5	5	1	0.3037	0.30000
75	0.30	0.5	0.5	0.5	0.5	0.5	10	10	5	5	1	0.3000	0.30000
75	0.30	0.5	0.5	0.5	0.5	0.5	7	5	5	4	4	0.3000	0.30000
75	0.30	0.5	0.5	0.5	0.5	0.5	7	5	5	5	3	0.3042	0.30000
75	0.30	0.5	0.5	0.5	0.5	0.5	8	5	5	4	3	0.3000	0.30000
75	0.30	0.5	0.5	0.5	0.5	0.5	7	5	4	4	4	0.3000	0.30000
100	0.50	0.5	0.5	0.5	0.5	0.5	10	10	5	5	5	0.5000	0.50000
100	0.50	0.5	0.5	0.5	0.5	0.5	10	5	4	5	5	0.5004	0.50000
100	0.50	0.5	0.5	0.5	0.5	0.5	10	5	5	5	5	0.5000	0.50000
100	0.50	0.5	0.5	0.5	0.5	0.5	10	5	5	5	2	0.5004	0.50000
20	0.05	0.5	0.5	0.5	0.5	0.5	20	10	5	4	5	0.0507	0.05000
20	0.05	0.5	0.5	0.5	0.5	0.5	20	10	5	5	5	0.0507	0.05000
20	0.05	0.5	0.5	0.5	0.5	0.5	10	10	10	5	1	0.0494	0.05000
20	0.05	0.5	0.5	0.5	0.5	0.5	11	10	10	5	1	0.0500	0.05000

Table 4.11. Parameter estimates and goodness-of-fit statistics for model (4.10).

Braded (deviance = 564.813) χ^2 = 468 (value/df = 1.8790)						
Braded χ^2 value = 565.8157, χ^2 = 456 (value/df = 1.8773)						
variable	parameter	estimate	std. err.	χ^2 stat	P-value	
α	β_0	0.0000	0.7856	0.7868	0.0001	
β	β_1	0.0000	0.0007	0.0000	0.0000	
ρ_1	β_2	-0.0000	0.0795	0.0796	0.0001	
ρ_2	β_3	0.0000	0.0126	0.0004	0.0000	
ρ_3	β_4	0.0000	0.0009	0.0000	0.0000	
ρ_4	β_5	0.0000	0.0001	0.0000	0.0000	
ρ_5	β_6	0.0000	0.0001	0.0000	0.0000	
ρ_6	β_7	-0.0000	0.0000	0.0000	0.0000	
ρ_7	β_8	0.0000	0.0000	0.0000	0.0000	
ρ_8	β_9	0.0000	0.0000	0.0000	0.0000	
ρ_9	β_{10}	0.0000	0.0000	0.0000	0.0000	
ρ_{10}	β_{11}	0.0000	0.0000	0.0000	0.0000	
ρ_{11}	β_{12}	0.0000	0.0000	0.0000	0.0000	
ρ_{12}	β_{13}	-0.0000	0.0000	0.0000	0.0000	
ρ_{13}	β_{14}	-0.0000	0.0000	0.0000	0.0000	
ρ_{14}	β_{15}	0.0000	0.0000	0.0000	0.0000	
ρ_{15}	β_{16}	-0.0000	0.0000	0.0000	0.0000	
ρ_{16}	β_{17}	0.0000	0.0000	0.0000	0.0000	
ρ_{17}	β_{18}	-0.0000	0.0000	0.0000	0.0000	
ρ_{18}	β_{19}	0.0000	0.0000	0.0000	0.0000	
ρ_{19}	β_{20}	-0.0000	0.0000	0.0000	0.0000	
ρ_{20}	β_{21}	0.0000	0.0000	0.0000	0.0000	
ρ_{21}	β_{22}	-0.0000	0.0000	0.0000	0.0000	
ρ_{22}	β_{23}	-0.0000	0.0000	0.0000	0.0000	
ρ_{23}	β_{24}	0.0000	0.0000	0.0000	0.0000	
ρ_{24}	β_{25}	0.0000	0.0000	0.0000	0.0000	
ρ_{25}	β_{26}	0.0000	0.0000	0.0000	0.0000	
ρ_{26}	β_{27}	0.0000	0.0000	0.0000	0.0000	
ρ_{27}	β_{28}	0.0000	0.0000	0.0000	0.0000	
ρ_{28}	β_{29}	0.0000	0.0000	0.0000	0.0000	
ρ_{29}	β_{30}	0.0000	0.0000	0.0000	0.0000	
ρ_{30}	β_{31}	0.0000	0.0000	0.0000	0.0000	
ρ_{31}	β_{32}	0.0000	0.0000	0.0000	0.0000	
ρ_{32}	β_{33}	0.0000	0.0000	0.0000	0.0000	
ρ_{33}	β_{34}	0.0000	0.0000	0.0000	0.0000	
ρ_{34}	β_{35}	0.0000	0.0000	0.0000	0.0000	
ρ_{35}	β_{36}	0.0000	0.0000	0.0000	0.0000	
ρ_{36}	β_{37}	0.0000	0.0000	0.0000	0.0000	
ρ_{37}	β_{38}	0.0000	0.0000	0.0000	0.0000	
ρ_{38}	β_{39}	0.0000	0.0000	0.0000	0.0000	
ρ_{39}	β_{40}	0.0000	0.0000	0.0000	0.0000	
ρ_{40}	β_{41}	0.0000	0.0000	0.0000	0.0000	
ρ_{41}	β_{42}	0.0000	0.0000	0.0000	0.0000	
ρ_{42}	β_{43}	0.0000	0.0000	0.0000	0.0000	
ρ_{43}	β_{44}	0.0000	0.0000	0.0000	0.0000	
ρ_{44}	β_{45}	0.0000	0.0000	0.0000	0.0000	
ρ_{45}	β_{46}	0.0000	0.0000	0.0000	0.0000	
ρ_{46}	β_{47}	0.0000	0.0000	0.0000	0.0000	
ρ_{47}	β_{48}	0.0000	0.0000	0.0000	0.0000	
ρ_{48}	β_{49}	0.0000	0.0000	0.0000	0.0000	
ρ_{49}	β_{50}	0.0000	0.0000	0.0000	0.0000	
ρ_{50}	β_{51}	0.0000	0.0000	0.0000	0.0000	
ρ_{51}	β_{52}	0.0000	0.0000	0.0000	0.0000	
ρ_{52}	β_{53}	0.0000	0.0000	0.0000	0.0000	
ρ_{53}	β_{54}	0.0000	0.0000	0.0000	0.0000	
ρ_{54}	β_{55}	0.0000	0.0000	0.0000	0.0000	
ρ_{55}	β_{56}	0.0000	0.0000	0.0000	0.0000	
ρ_{56}	β_{57}	0.0000	0.0000	0.0000	0.0000	
ρ_{57}	β_{58}	0.0000	0.0000	0.0000	0.0000	
ρ_{58}	β_{59}	0.0000	0.0000	0.0000	0.0000	
ρ_{59}	β_{60}	0.0000	0.0000	0.0000	0.0000	
ρ_{60}	β_{61}	0.0000	0.0000	0.0000	0.0000	
ρ_{61}	β_{62}	0.0000	0.0000	0.0000	0.0000	
ρ_{62}	β_{63}	0.0000	0.0000	0.0000	0.0000	
ρ_{63}	β_{64}	0.0000	0.0000	0.0000	0.0000	
ρ_{64}	β_{65}	0.0000	0.0000	0.0000	0.0000	
ρ_{65}	β_{66}	0.0000	0.0000	0.0000	0.0000	
ρ_{66}	β_{67}	0.0000	0.0000	0.0000	0.0000	
ρ_{67}	β_{68}	0.0000	0.0000	0.0000	0.0000	
ρ_{68}	β_{69}	0.0000	0.0000	0.0000	0.0000	
ρ_{69}	β_{70}	0.0000	0.0000	0.0000	0.0000	
ρ_{70}	β_{71}	0.0000	0.0000	0.0000	0.0000	
ρ_{71}	β_{72}	0.0000	0.0000	0.0000	0.0000	
ρ_{72}	β_{73}	0.0000	0.0000	0.0000	0.0000	
ρ_{73}	β_{74}	0.0000	0.0000	0.0000	0.0000	
ρ_{74}	β_{75}	0.0000	0.0000	0.0000	0.0000	
ρ_{75}	β_{76}	0.0000	0.0000	0.0000	0.0000	
ρ_{76}	β_{77}	0.0000	0.0000	0.0000	0.0000	
ρ_{77}	β_{78}	0.0000	0.0000	0.0000	0.0000	
ρ_{78}	β_{79}	0.0000	0.0000	0.0000	0.0000	
ρ_{79}	β_{80}	0.0000	0.0000	0.0000	0.0000	
ρ_{80}	β_{81}	0.0000	0.0000	0.0000	0.0000	
ρ_{81}	β_{82}	0.0000	0.0000	0.0000	0.0000	
ρ_{82}	β_{83}	0.0000	0.0000	0.0000	0.0000	
ρ_{83}	β_{84}	0.0000	0.0000	0.0000	0.0000	
ρ_{84}	β_{85}	0.0000	0.0000	0.0000	0.0000	
ρ_{85}	β_{86}	0.0000	0.0000	0.0000	0.0000	
ρ_{86}	β_{87}	0.0000	0.0000	0.0000	0.0000	
ρ_{87}	β_{88}	0.0000	0.0000	0.0000	0.0000	
ρ_{88}	β_{89}	0.0000	0.0000	0.0000	0.0000	
ρ_{89}	β_{90}	0.0000	0.0000	0.0000	0.0000	
ρ_{90}	β_{91}	0.0000	0.0000	0.0000	0.0000	
ρ_{91}	β_{92}	0.0000	0.0000	0.0000	0.0000	
ρ_{92}	β_{93}	0.0000	0.0000	0.0000	0.0000	
ρ_{93}	β_{94}	0.0000	0.0000	0.0000	0.0000	
ρ_{94}	β_{95}	0.0000	0.0000	0.0000	0.0000	
ρ_{95}	β_{96}	0.0000	0.0000	0.0000	0.0000	
ρ_{96}	β_{97}	0.0000	0.0000	0.0000	0.0000	
ρ_{97}	β_{98}	0.0000	0.0000	0.0000	0.0000	
ρ_{98}	β_{99}	0.0000	0.0000	0.0000	0.0000	
ρ_{99}	β_{100}	0.0000	0.0000	0.0000	0.0000	

Table 4.15: The estimated power, $\pi(\alpha)$, and its maximum and average differences with π_{LS} , the F^{LS} power value at a given point of the bootstrap design.

F	$\log(P)$	δ	β_1	β_2	β_3	β_4	β_5	β_6	π	$\max \pi - \pi_{\text{LS}} $	$\text{ave} \pi - \pi_{\text{LS}} $
D0	0.00000	0.60	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.1	0.0	0.0	0.00000	-0.00000	0.00000
			0.0	0.0	0.1	0.0	0.0	0.0	0.00000	0.00000	-0.00000
			0.0	0.0	0.1	0.1	0.0	0.0	0.00000	-0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.1	0.0	0.0	0.00000	-0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	-0.00000	-0.00000
T0	-1.30750	0.40	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	-0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	-0.00000
			0.0	0.0	0.0	0.1	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
T0	-2.30000	0.40	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	-0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	-0.00000	-0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	-0.00000
			0.0	0.0	0.0	0.1	0.0	0.0	0.00000	-0.00000	-0.00000
D0	0.00000	0.80	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
D0	0.00000	0.80	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	-0.00000	-0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.0	0.00000	-0.00000	-0.00000

Table 4.14: —continued. The estimated power, $\pi(\alpha)$, and its maximum and average differences with π_0 , the α^2 power value in a given point, of the Das-Balakrishnan design

N	$\log(N)$	β	β_1	β_2	β_3	β_4	β_5	π	$\max \pi - \pi_0 $	$\text{ave} \pi - \pi_0 $
75	4.87549	0.05	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.4	0.1	0.0	0.0	0.0	0.00000	-0.00000	-0.00000
75	4.87549	0.10	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.1	0.0	0.0	0.0	0.00000	-0.00000	-0.00000
75	4.87549	0.20	0.0	0.0	0.0	0.0	0.0	0.00000	-0.00000	-0.00000
			0.0	0.1	0.0	0.0	0.0	0.00000	-0.00000	-0.00000
75	4.87549	0.30	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
75	4.87549	0.40	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
75	4.87549	0.05	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
75	4.87549	0.10	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
75	4.87549	0.20	0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000
			0.0	0.0	0.0	0.0	0.0	0.00000	0.00000	0.00000

Table 4.11. —continued. The estimated power, $\hat{\pi}(\mathbf{x})$, and its maximum and average differences with π_0 , the β^0 power value at a given point of the FactorialDesign design

β	$\log(\beta^0)$	$\mathbf{x}$	β_1	β_2	β_3	β_4	β_5	β_6	π	$\max \hat{\pi} - \pi_0 $	ave. diff.
(5)	3.0000	0.50	0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.0000	-0.0000
			0.0	0.1	0.0	0.1	0.0	0.0	0.1000	-0.0779	-0.0261
			0.0	0.1	0.0	0.0	0.0	0.0	0.0000	0.0000	0.0000
			0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.0000	0.0000
(6)	3.0000	0.30	0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.0000	0.0000
			0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.1000	-0.0000
			0.0	0.0	0.0	0.0	0.0	0.0	0.0000	0.0000	0.0000
			0.0	0.0	0.0	0.0	0.0	0.0	0.0000	0.0000	0.0000
(7)	3.0000	0.10	0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.0000	0.0000
			0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.0000	-0.0000
			0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.0000	0.0000
			0.0	0.0	0.0	0.0	0.0	0.0	0.0000	-0.0000	0.0000

For each design in Table 4.10, the exact power is calculated using Davies' algorithm. The predicted power is also obtained using the model (4.28). These values along with its differences are listed in Table 4.11. From this table, we can see that the prediction capability of the model (4.28) is still good even at points outside the region $\mathcal{C}$ in (4.26).

4.4.1. Continued analysis of mc200 (4.28)

Model (4.28) with parameter estimation in Table 4.14 can be used to determine confidence on $\pi := (\pi_0, \pi_1, \dots, \pi_7)'$ that results in large values of π . We recall that $\pi_0 = \log(\beta^0)$, $\pi_1 = \beta_1$, $\pi_2 = \beta_2$, $\pi_3 = \beta_3$, $\pi_4 = \beta_4$, $\pi_5 = \beta_5$, $\pi_6 = \beta_6$, $\pi_7 = \beta_7$. Let a_k and b_k denote the lower and upper bounds, respectively, on π_k as described at the region $\mathcal{C}$ in (4.16) ($\pi = 1, 2, \dots, 7$). It is convenient here to code the π_k using the variable

Table 4.16: Generated designs for the two-way model with $k = 3$ to check the prediction capability of model (4.24) - $\hat{\mu}$, denotes the actual value of μ for a n -generated design

n	$\hat{\mu}$	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}	μ_{12}
15	0.1	0.2	0.8	0.7	0.3	0.1	10	3	1	1	1	0.4888	
20	0.1	0.2	0.8	0.7	0.3	0.1	8	3	3	3	3	0.7000	
33	1.0	0.2	0.8	0.7	0.3	0.1	5	3	3	3	3	1.0000	
100	0.1	0.7	0.7	0.7	0.7	0.7	20	20	10	17	10	0.6478	
100	0.3	0.5	0.5	0.5	0.5	0.5	20	20	20	15	3	0.6624	
100	1.0	0.7	0.7	0.7	0.7	0.7	20	20	20	20	20	1.0000	

Table 4.17: Prediction capability of model (4.24) for designs in Table 4.16

μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ	$\hat{\mu}$	$\mu - \hat{\mu}$
0.2	0.8	0.7	0.3	0.1	10	3	1	1	1	0.12437	0.39999	0.11557
0.2	0.8	0.7	0.3	0.1	8	3	3	3	3	0.21550	0.70073	-0.20213
0.2	0.8	0.7	0.3	0.1	5	3	3	3	3	0.19033	0.70000	-0.20067
0.7	0.7	0.7	0.7	0.7	20	20	10	17	10	0.65495	0.64780	0.00715
0.5	0.5	0.5	0.5	0.5	20	20	20	15	3	0.65033	0.66624	-0.00091
0.7	0.7	0.7	0.7	0.7	20	20	20	20	20	0.66433	0.66624	-0.00091

a_i such that $-1 < \frac{a_i}{b_i} < 1$, $i = 1, 2, \dots, T$. This is accomplished by considering the transformation

$$a_i = \frac{2a_i - (a_i + b_i)}{b_i - a_i}, \quad i = 1, 2, \dots, T, \quad (4.18)$$

or then have

$$x = Gw + d, \quad (4.19)$$

where $x = (x_1, x_2, \dots, x_T) \in \mathbb{R}^T$, $G = G_0^T, \left\{ \frac{1}{b_i - a_i} \right\}$, and the i^{th} element of d is $d_i = \frac{2(a_i + b_i)}{b_i - a_i}$, $i = 1, 2, \dots, T$. Using x in place of a in the second-degree model (4.15), we get

$$\zeta = \gamma_0 + x'\gamma + x'\Gamma x, \quad (4.20)$$

where $\gamma_0 = \beta_0 - d'\bar{Q}^{-1}\beta + d'\bar{Q}^{-1}\bar{B}\bar{Q}^{-1}d$, $\gamma = G^{-1}\beta - (G^{-1}\bar{B}\bar{Q}^{-1}d$ and $\Gamma = \bar{Q}^{-1}\bar{B}\bar{Q}^{-1}$. The estimated value of ζ is

$$\hat{\zeta} = \gamma_0 + x'\hat{\gamma} + x'\hat{\Gamma}x, \quad (4.21)$$

where γ_0 , $\hat{\gamma}$ and $\hat{\Gamma}$ are obtained by using the maximum likelihood estimates of β_0 , β and $\bar{B}$. From (4.21), the predicted value of v at a point a in the region C_n , where

$$C_n = \{a \mid -1 \leq a_i \leq 1, \quad i = 1, 2, \dots, T\}, \quad (4.22)$$

is given by

$$\hat{v}(a) = \frac{1}{1 + \exp(\hat{\zeta} + x'\hat{\gamma} + x'\hat{\Gamma}x)}. \quad (4.23)$$

In order to study the behavior of $\hat{v}(a)$, it would be convenient to express the linear predictor $\hat{\zeta}$ in (4.21) in terms of its canonical form, namely,

$$\hat{\zeta} = \zeta_0 + \sum_{i=1}^T \lambda_i W_i^2, \quad (4.24)$$

where ζ_0 is the value of $\hat{\zeta}$ at $a_0 = -\frac{1}{2}E^{-1}\bar{B}$, the stationary point of $\hat{\zeta}$, $w_1, w_2, \dots, w_T$ are the eigenvalues of $\hat{\Gamma}$, and W_i is the i^{th} element of $\hat{W} = \hat{P}(a - a_0)$. Here, $\hat{P}$ is an

orthogonal matrix that diagonalizes $\mathbf{F}$ that is $\mathbf{F} = \mathbf{P}\mathbf{A}\mathbf{P}^T$, where $\mathbf{A}$ is the diagonal matrix of eigenvalues of $\mathbf{F}$ (see Kline and Cornell, 1998, Section 3.3.4). Note that the columns of $\mathbf{P}$ are orthonormal eigenvectors of $\mathbf{F}$ corresponding to $\alpha_1, \alpha_2, \dots, \alpha_7$. Using the parameter estimates in Table 4.34, we have

$$\hat{\alpha}_1 = 14.987$$

$$\hat{\alpha}_2 = (-2.829, 2.982, -0.263, -0.408, -0.085, -12.140, 1.004)^T$$

$$\mathbf{P}^T = \begin{bmatrix} -0.027 & -0.309 & -0.732 & 0.399 & 0.352 & 0.155 & 0.060 \\ -0.035 & 0.044 & 0.028 & 0.607 & -0.704 & -0.408 & 0.178 \\ -0.136 & 0.490 & -0.061 & 0.435 & 0.497 & -0.037 & -0.401 \\ 0.411 & 0.374 & -0.294 & 0.249 & 0.375 & 0.039 & 0.079 \\ 0.296 & -0.468 & 0.320 & 1.367 & 0.230 & -0.343 & 0.095 \\ 0.004 & -0.160 & 0.494 & 0.461 & 0.294 & 0.002 & -0.276 \\ 0.038 & -0.036 & -0.390 & -0.117 & -0.256 & -0.236 & -0.451 \end{bmatrix}$$

$$\mathbf{A} = \text{diag} \{ 2.375, 0.023, 0.421, -0.403, 0.209, -0.041, 0.000 \} \quad (4.36)$$

We note that α_1 falls outside the region C_1 in (4.26). Also, note the eigenvalues in (4.26) have variable signs: α_2 is a saddle point.

The main advantage of the canonical equation (4.35) is the ability to study the behavior of x along the principal W_i -axes, that is, along axes that are determined by the rows of $\mathbf{P}^T$. This is done as follows: since α_1 and α_2 are negative (see (4.36)), a move away from x_0 along the W_1 -axis or the W_2 -axis in either direction results in a decrease in f and hence an increase in x since $x = \frac{1}{1.0000}$. The maximum in x is most likely along the W_1 -axis since $|\alpha_1|$ is larger than $|\alpha_2|$. On the other hand, a move away from x_0 along each of the W_i -axis, $i = 3, 4, 5, 6, 7$ in either direction results in no increase in f and hence a decrease in x . The decrease in x is most apparent

along the W_1 axis since α_1 is larger in terms of absolute value, than all the other eigenvalues.

Since the stationary point α_1 is a saddle point and falls inside the region C_1 in (4.23), we should determine the absolute optima (maximum and minimum) of $\bar{t}$ within C_1 , or equivalently, the region C in (4.26). This work will be addressed in the next section.

4.2.3 Global value of $\bar{t}$

Let us write the linear predictor in (4.25) as

$$\xi = \mathcal{F}(\mathbf{x})\boldsymbol{\beta}, \quad (4.27)$$

where $\boldsymbol{\beta}$ is a vector of parameters and $\mathcal{F}(\mathbf{x})$ is a vector of powers and cross products of powers of the elements of $\mathbf{x}$ up to degree 2 such that $\mathcal{F}(\mathbf{x})\boldsymbol{\beta} = \beta_0 + \mathbf{x}'\boldsymbol{\beta} + \mathbf{x}'\mathbf{B}\mathbf{x}$. By evaluating $\bar{t}$ at each point of the first-order design (mentioned earlier in Section 4.2.2) we get the model

$$\xi = \mathbf{X}\boldsymbol{\beta}, \quad (4.28)$$

where $\mathbf{X} = [\mathcal{F}(\mathbf{x}_1) \cdots \mathcal{F}(\mathbf{x}_n)]$, $[\mathcal{F}(\mathbf{x}_i)]' = \mathbf{1} + \sum_{j=1}^m m_j = 500$ with m_j being the number of replicates at the i^{th} design point, $i = 1, 2, \dots, 50$, and ξ is the $n \times 1$ vector of corresponding values of ξ in (4.27). The predicted value of $\bar{t}$ at a point $\mathbf{x}$ in the region C in (4.26), denoted by $\hat{t}(\mathbf{x})$ is $\hat{t}(\mathbf{x}) = \mathcal{F}(\mathbf{x})\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is such that $\mathcal{F}(\mathbf{x})\hat{\boldsymbol{\beta}} = \hat{\beta}_0 + \mathbf{x}'\hat{\boldsymbol{\beta}} + \mathbf{x}'\hat{\mathbf{B}}\mathbf{x}$. The variance of $\hat{t}(\mathbf{x})$ is approximately given by

$$\text{Var}[\hat{t}(\mathbf{x})] = \mathcal{F}(\mathbf{x})\mathbf{X}'\mathbf{X}^{-1}\mathbf{X}\mathcal{F}(\mathbf{x}), \quad (4.29)$$

where $\mathbf{X}$ is a diagonal matrix whose i^{th} diagonal element is $\frac{1}{n}[\mathcal{F}(\mathbf{x}_i)]'$, $i = 1, \dots, n$ where $\mathcal{F}(\mathbf{x}_i) = \log(\mu_i)$ with μ_i being the mean of the distribution of ω in (4.25) at the i^{th} design point, and $\mathcal{F}(\mathbf{x}_i)$ is the first derivative of $\bar{t}$ evaluated at μ_i . Furthermore, by use of the gamma distribution for ω , the variance of this distribution at

the i^{th} point $\mathbf{u}$ given by $\frac{d\mathbf{u}}{dv}$ where v is a scalar parameter. An estimate $\hat{\mathbf{u}}$ of $\mathbf{u}$ is obtained via an iteratively weighted least-squares procedure. Hence, an estimate of the variance of $\hat{r}(\mathbf{u}) = \frac{1}{\text{tr}(\hat{\mathbf{C}}(\mathbf{u}))}$ is approximately given by

$$\widehat{\text{Var}}(\hat{r}(\mathbf{u})) = \frac{\text{tr}(\hat{\mathbf{C}}'(\mathbf{u})\hat{\mathbf{C}})}{\{1 + \text{tr}(\hat{\mathbf{C}}'(\mathbf{u})\hat{\mathbf{C}})\}^2} \hat{\mathbf{r}}'(\mathbf{u})(\mathbf{X}\hat{\mathbf{A}}^{-1}\mathbf{X}')^{-1}\hat{\mathbf{r}}(\mathbf{u}) \quad (3.46)$$

Let us now divide $\hat{r}(\mathbf{u})$ by its standard deviation to get

$$\hat{r}(\mathbf{u}) = \frac{\hat{r}(\mathbf{u})}{\{\widehat{\text{Var}}(\hat{r}(\mathbf{u}))\}^{1/2}} \quad (3.47)$$

The function $\hat{r}(\mathbf{u})$ is a scaled version of $\hat{r}(\mathbf{u})$ and is the one to be optimized over the region C . This is accomplished numerically through a grid search over C . It was found that $\hat{r}(\mathbf{u})$ attained its maximum value, namely 5.174, at the point $\mathbf{u}_{\max}$ whose coordinates are $\alpha_1 = \log(N)$ with $N = 25$, $\alpha_2 = \beta = 0.3$, $\alpha_3 = \rho_1 = 0.1$, $\alpha_4 = \rho_2 = 0.9$, $\alpha_5 = \beta_1 = 0.9$, $\alpha_6 = \rho_3 = 0.9$ and $\alpha_7 = \rho_4 = 0.9$. The corresponding value of $\hat{r}(\mathbf{u})$ is 0.004 with a standard error approximately equal to 0.0003. The minimum value of $\hat{r}(\mathbf{u})$ over C is 1.54563, and is attained at the point $\mathbf{u}_{\min}$ whose coordinates are $\alpha_1 = \log(N)$, with $N = 75$, $\alpha_2 = \beta = 1.0$, $\alpha_3 = \rho_1 = 0.9$, $\alpha_4 = \rho_2 = 0.9$, $\alpha_5 = \beta_1 = 0.9$, $\alpha_6 = \rho_3 = 0.9$ and $\alpha_7 = \rho_4 = 0.9$. The corresponding value of $\hat{r}(\mathbf{u})$ is 0.0004 with a standard error approximately equal to 0.00001. We note that both optima occur on the boundary of the region C . We also note that at $\mathbf{u}_{\max}$ the values of N, β , and ρ_4 are equal to their lower bounds in C whereas the values of ρ_1, ρ_2, ρ_3 , and ρ_4 are equal to their respective upper bounds. On the other hand, the values of N, β , and ρ_1 at $\mathbf{u}_{\min}$ coincide with their upper bounds, and ρ_2, ρ_3, ρ_3 and ρ_4 maintain the same values as in $\mathbf{u}_{\max}$. Therefore the maximum points occur when the errors are homogeneous (recall that $\rho_4 = \frac{\sigma_4^2}{\sigma_2^2}$) the data set is balanced (since $\beta = 1$) and the value of N is largest within the region C in (3.26).

4.3.4. The power trace

In this section we show how to monitor the increase in the power values as we move from $\mathbf{u}_{\min}$ to $\mathbf{u}_{\max}$. This is accomplished by using plots of the so-called power trace, which we now describe.

Using the coding convention of formula (3.33), $\mathbf{u}_{\min}$ and $\mathbf{u}_{\max}$ can be written as $\mathbf{u}_{\min} = [-1, -1, -1, 1, 1, 1, 1]^T$, $\mathbf{u}_{\max} = [1, 1, 1, 1, 1, 1, 1]^T$, respectively. Let us introduce the s -coordinates system so that its origin coincides with $\mathbf{u}_{\min}$. This is accomplished by the transformation $\mathbf{u} = \mathbf{s} + \mathbf{u}_{\min}$, where $\mathbf{s} = [s_1, s_2, \dots, s_7]^T$. From (4.33), the relationship between $\mathbf{u}$ and $\mathbf{s}$ is

$$\mathbf{u} = \mathbf{U}^{-1}(\mathbf{s} - \mathbf{d} + \mathbf{u}_{\min}) \quad (4.42)$$

The region G_u in (3.33) can then be expressed as

$$G_u = \{\mathbf{s} \mid 0 \leq s_i \leq 2, \quad i = 1, 3, 4, \quad -2 \leq s_i \leq 2, \quad i = 2, 5, 6, 7\}, \quad (4.43)$$

where s_i is the i^{th} element of $\mathbf{s}$, $i = 1, 2, \dots, 7$. The points $\mathbf{u}_{\min}$ and $\mathbf{u}_{\max}$ correspond to $\mathbf{s}_{\min} = [0, 0, 0, 0, 0, 0, 0]^T$, $\mathbf{s}_{\max} = [2, 2, 2, 2, 0, 0, 0]^T$, respectively. In addition, the coded predicted power function $q(\mathbf{u})$ in (4.41) can be written as $\hat{p}(\mathbf{s})$, where by (4.42),

$$\hat{p}(\mathbf{s}) = q(\mathbf{U}^{-1}(\mathbf{s} - \mathbf{d} + \mathbf{u}_{\min}))$$

In order to study the increase in power, that is, in the values of $\hat{p}(\mathbf{s})$ as we move from $\mathbf{u}_{\min}$ to $\mathbf{u}_{\max}$ within the region G_u , we propose to find maximum values of $\hat{p}(\mathbf{s})$ on the surfaces of concentric hyperspheres of varying radii centered at $\mathbf{s}_{\min}$. This is analogous to the method of edge analysis commonly used in response surface methodology [see, for example, Khuri and Cornell, 1996, Section 5.7]. In our case, maximumness is confined to the portions of the hyperspheres contained inside G_u . This process is facilitated by expressing $s_1, s_2, \dots, s_7$ in terms of the spherical coordinates,

$\mathbf{u} = (u_1, u_2, \dots, u_8)$, such that

$$u_1 = \cos \theta_1$$

$$u_2 = \cos \theta_2 \sin \theta_1$$

$$u_3 = \cos \theta_3 \sin \theta_2 \sin \theta_1$$

$$u_4 = \cos \theta_4 \sin \theta_3 \sin \theta_2 \sin \theta_1$$

$$u_5 = \cos \theta_5 \sin \theta_4 \sin \theta_3 \sin \theta_2 \sin \theta_1$$

$$u_6 = \cos \theta_6 \sin \theta_5 \sin \theta_4 \sin \theta_3 \sin \theta_2 \sin \theta_1$$

$$u_7 = \cos \theta_7 \sin \theta_6 \sin \theta_5 \sin \theta_4 \sin \theta_3 \sin \theta_2 \sin \theta_1$$

where $0 \leq \theta_i \leq \pi/2$, $0 \leq \theta_8 \leq \pi$, $i = 1, 2, 3, 4, 5, 6, 7$ and θ_8 is the distance of $\mathbf{u}_{\text{max}}$ from $\mathbf{u}_{\text{min}}$.

For s given values, s_i points are generated on the surface of the hypersphere $T_i = \{\mathbf{u} | \mathbf{u}^T \mathbf{u} = r^2\}$ by selecting $\theta_1, \theta_2, \dots, \theta_8$ at random, using independent uniform distributions. Points that do not fall inside $\mathcal{C}_i$ are discarded. The remaining points are used to estimate the maximum of $p(\mathbf{u})$ on T_i . Let such a maximum be denoted by $p_{\text{max}}(s)$. The point at which the maximum is attained is denoted by $\mathbf{u}_{\text{max}}^{(s)} = (u_1^{(s)}, u_2^{(s)}, \dots, u_8^{(s)})$. The next step is to plot $p_{\text{max}}(s)$ as well as $u_i^{(s)}$ against s , $i = 1, 2, \dots, 7$. These plots which we refer to as power trace plots, are shown in Figure 4.7. Note that Figure 4.7(a) gives a plot of the scaled predicted power, $\hat{p}(\mathbf{x})$ in (3.6), against s and Figure 4.7(b) is for the corresponding predicted power, $p(\mathbf{x})$ in (3.8) against s . Also, Figure 4.7(c) gives plots of $u_i^{(s)}$ against s where $u_i^{(s)}$ is the i^{th} element of $\mathbf{u}_{\text{max}}^{(s)} = \mathbf{u}_{\text{max}}^{(s)} + \mathbf{u}_{\text{min}}$. The reason for using $u_i^{(s)}$ instead of $u_i^{(s)2}$ is due to the fact that $-1 \leq u_i^{(s)} \leq 1$ for $i = 1, 2, \dots, 7$, which facilitates the interpretation of the trace plots of $u_i^{(s)}$. These plots correspond to $\log(N), \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ and θ_7 , respectively.

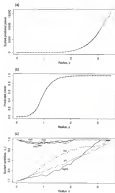


Figure 6.7: System input plots

From Figure 4.7(a) it can be easily seen that μ_1, μ_2, μ_3 , and μ_4 meet their effect on the power. On the other hand, N, σ_1 , and σ_2 in particular contribute significantly to the increase in power. Thus, the power is more sensitive to changes in N, σ_1 , and σ_2 . Note that for $i \leq 2, 4, \mu_i$ is most influential, followed by σ_1 and N . Hence, the power is most sensitive with respect to μ_i . We may recall that μ_4 is associated with the largest group size n_4 since the n_i values in Table 4.12 were arranged in descending order. Figure 4.7(c) shows that the values of μ_i are smaller than those of μ_4 for $i = 1, 2, 3, 4, 5$, hence σ_i^2 is larger than σ_4^2 for $i = 1, 2, 3, 4, 5$. We conclude that assigning the largest number of experimental units to the most variable group makes an variance most influential with regard to the power of the F -test. Thus, assigning the largest sample size to the group having largest variance can produce high power values provided that the largest variance is not too large, noting that if the largest variance is small, the other variances are also small.

4.3.7. Concluding remarks

The modeling of the power of the F -test against $N, \sigma_1, \sigma_2, \dots, \mu_5$ enables one to clearly assess the effects of these parameters on the power. Response surface techniques can be successfully used for this purpose. In particular, the canonical analysis in Section 4.2.4 is utilized to determine the nature of the stationary point of the fitted surface, and to identify principal axes along which the power increases or decreases. Furthermore, plots of the power trace in Section 4.2.4 provide a graphical depiction of the behavior in the power. These plots are also useful in detecting those main variances that are influential contributors to the increase in the power.

4.4 The Coverage Probability of a Confidence Interval

4.4.1 Introduction

Confidence intervals for variance components are of interest in any study which deals with random or mixed effects models. Many papers have been written on the subject of confidence intervals for variance components, functions of variance components, and simultaneous confidence intervals for a variety of models. For a wide coverage of this subject area, we refer to Burdick and Graybill (1983).

An exact confidence interval as a function of variance components does not always exist. This is true even for the one-way random model with two variance components, σ_1^2 and σ_2^2 . When the data set is balanced, there are exact confidence intervals on σ_1^2 , the ratio $\frac{\sigma_1^2}{\sigma_2^2}$ and the intraclass correlation coefficient, $\rho = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$ (see Burdick and Graybill 1983 Ch. 4). However, there does not exist an exact confidence interval on σ_2^2 as Thomas and Madowitz (1978, p. 363) indicated. Several studies were made to construct an approximate confidence interval on σ_2^2 (see, for example, Bartlett, 1942; Brown, 1946; Palmer 1947; Cochran, 1954; Fien, 1976; Margolis, 1984; Santaferruta, 1941; Tolley, 1961; Welch, 1958; Williams, 1952; to name just a few). Burdick (1974) compared these approximate procedures by using Monte Carlo simulation, to determine the actual probability coverage for each procedure. His study showed that two procedures provided a coverage probability close to the nominal level. One of these procedures was independently developed by Tolley (1961) and Williams (1962). The other procedure was independently developed by Margolis (1984) and Palmer (1982).

For the unbalanced one-way random model, only σ_1^2 has an exact confidence interval. For other parameters of interest, several approximate confidence intervals have

been proposed. For example, for σ^2_{ϵ} , the work of Thomas and Holguin (1979), Burchard and Schimay (1995), and El-Derdori (2004) are cited. Graybill and Wang (1962) considered a balanced K -factor nested random model. They proposed a general procedure for constructing a confidence interval as a linear combination of variance components based on K independently distributed chi-squared random variables. Burdick and Graybill (1964) proposed a modification of the Graybill-Wang procedure that considered the unbalancedness of the design. Their modification was for $K = 2$ which is for the unbalanced one-way random model. Burdick, Shapard and Graybill (1965), Grosser and Wells (1966), and Mee, Shapard, and Tracy (1966) obtained confidence intervals on the ratio of variance components, and the intraclass correlation coefficient. For models other than the one-way random model, Katsouris and Graybill (1967) considered the unbalanced two-way random cross-classification with interaction model. Graybill and Wang (1976) and Wang and Graybill (1994) focused on the balanced two-fold nested random model. Ren, Graybill, and Tsai (2001), Bernardino, Burdick and Block (1993), Bernardino and Burdick (2002) considered the unbalanced two-fold nested random model, and Reynolds and Pineda (1981) dealt with an unbalanced mixed linear model. For a survey on confidence intervals up to 1994 see Klein and Rubin (1995).

Even though many papers were written on confidence intervals for variance components, there is no study devoted solely to the effect of imbalance on the coverage probability of confidence intervals. The coverage probability of a certain confidence interval depends not only on the variance components, but also on the design used. Researchers have used particular designs ranging from extremely unbalanced to nearly balanced in order to evaluate the performance of their proposed methods.

In Section 4.1, we shall investigate the effects of imbalance and variance components on the coverage probabilities of several confidence intervals. For the unbalanced, one-way random model, four approximate methods of confidence interval on σ_e^2 are considered. These are given in Section 4.1.1. In Section 4.1.2, we propose a procedure for modeling the coverage probability associated with each of these methods in terms of factors that influence the coverage probability. Section 4.1.3 is devoted to comparing these methods. Concluding remarks are given in Section 4.1.4.

4.1.2. Confidence intervals on σ_e^2 for the one-way model

Consider the one-way unbalanced random model (3.1) with its vector form in (3.2). The sums of squares for the treatment effect, $y'Qy$, and the error term, $y'Ry$, where Q and R are defined in (3.7) and (3.8), respectively, are independent. When the design is balanced (i.e., $n_1 = n_2 = \cdots = n_k = n$), these sums of squares are scaled chi-squared variables with scales $n(\sigma_e^2 + \sigma_\tau^2)$ and σ_e^2 , respectively. However, when n_k 's are not equal, $y'Qy$ has a scaled chi-squared distribution, if and only if $\sigma_e^2 = 0$. If it is known that $\sigma_e^2 = 0$, then it may be appropriate to treat $y'Qy$ approximately as a scaled chi-squared variable. This motivates the use of an approximate confidence interval on σ_e^2 for the balanced model by replacing α with an appropriate value.

Modified Large Sample (MLS) Confidence Interval

Balducci and Graybill (1983, p. 34, p. 75) formulated a confidence interval based on the above-mentioned strategy. The resulting approximate two-sided $(1 - \alpha)(100\%)$ confidence interval on σ_e^2 is

$$\left[\frac{y'Sy - y'Ty - \alpha\sqrt{V_y}}{n_k}, \frac{y'Sy - y'Ty + \alpha\sqrt{V_y}}{n_k} \right], \quad (4.4)$$

where

$$\mathbf{y}_n = \frac{1}{n-1}(\mathbf{y} - \frac{1}{n}\sum_{i=1}^n \mathbf{y}_i^2)$$

$$\mathbf{W}^2\mathbf{S}_n = \frac{1}{n-1}\mathbf{y}'^2\mathbf{Q}\mathbf{y}$$

$$\mathbf{W}\mathbf{S}_n = \frac{1}{n-1}\mathbf{y}'^2\mathbf{S}\mathbf{y}_n$$

$$\mathbf{V}_1 = \mathbf{C}_1^2\mathbf{W}\mathbf{S}_n^2 + \mathbf{R}_1^2\mathbf{M}\mathbf{S}_n^2 + \mathbf{C}_{12}\mathbf{W}\mathbf{S}_n\mathbf{W}'\mathbf{S}_n$$

$$\mathbf{V}_2 = \mathbf{R}_2^2\mathbf{M}\mathbf{S}_n^2 + \mathbf{C}_2^2\mathbf{M}\mathbf{S}_n^2 + \mathbf{R}_{22}\mathbf{M}\mathbf{S}_n\mathbf{M}'\mathbf{S}_n$$

$$\hat{\alpha}_1 = 1 - \frac{1}{F_{\frac{\alpha}{2}}(k-1, n)},$$

$$\hat{\alpha}_n = 1 - \frac{1}{F_{\frac{\alpha}{2}}(N-k, n)},$$

$$\hat{\beta}_1 = \frac{1}{F_{1-\frac{\alpha}{2}}(N-1, n)} - 1,$$

$$\hat{\beta}_n = \frac{1}{F_{1-\frac{\alpha}{2}}(N-k, n)} - 1,$$

$$G_{12} = \frac{[F_{\frac{\alpha}{2}}(k-1, N-k) - 1]^2 - (G_1 F_{\frac{\alpha}{2}}(k-1, N-k)^2 - \hat{\alpha}_1^2)}{F_{\frac{\alpha}{2}}(k-1, N-k)}, \text{ and}$$

$$R_{12} = \frac{[1 - F_{1-\frac{\alpha}{2}}(k-1, N-k)]^2 - (R_1 F_{1-\frac{\alpha}{2}}(k-1, N-k)^2 - \hat{\beta}_1^2)}{F_{1-\frac{\alpha}{2}}(k-1, N-k)}.$$

Here $\alpha_1^2 \geq 0$, the lower bound is set to zero if $\frac{G_1^2}{G_{12}} < F_{\frac{\alpha}{2}}(k-1, N-k)$. Also, the upper bound is defined to be zero if $\frac{R_1^2}{R_{12}} < F_{1-\frac{\alpha}{2}}(k-1, N-k)$. Here, $F_2(v_1, v_2)$ denotes the upper 100α percentile of an F -distribution with v_1 and v_2 degrees of freedom. We shall refer to formula (4.44) as the MLE confidence interval on α_1^2 .

Thomas-Pollard's (TP) Confidence Interval

The unweighted sum of squares associated with the treatment means is calculated

one-way random model is $\sum_{i=1}^k \bar{y}_i^2 - k\bar{y}^2$ where $\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$ and $\bar{y} = \frac{1}{N} \sum_{i=1}^k n_i \bar{y}_i$. It is noted that

$$s_y^2 = \frac{1}{k-1} \sum_{i=1}^k (\bar{y}_i - \bar{y})^2 \quad (4.41)$$

is the sample variance for the treatment means. Also note that $E(s_y^2) = \sigma_1^2 + \frac{1}{k} \sigma_2^2$, where $\sigma_2 = \sqrt{\sum_{i=1}^k \frac{1}{n_i}}$ is the harmonic mean of the n_i values (see Garfield and Graybill 1984, p. 132). Thomas and Holguin (1978) showed that the moment generating function of their W random variable, which is $W = \frac{E(s_y^2) - s_y^2}{E(s_y^2)}$, approaches that of the equated random variable as all n_i values approach a constant, or as all $n_i \rightarrow \infty$ or as $\eta = \frac{N}{k} \rightarrow \infty$. They also provided simulation results that W is well approximated by a chi-squared random variable when $\eta \geq 3.25$ and the design is not extremely unbalanced. Using W , they derived formulas for constructing confidence intervals on σ_1^2 , $\frac{\sigma_1^2}{\sigma_2^2}$, and $\frac{\sigma_1^2}{E(s_y^2)}$. More specifically, an approximate two-sided $(1 - \alpha)(100)\%$ confidence interval on σ_1^2 is obtained by replacing $M\bar{y}_i$ with $n_i s_y^2$ and n with n_k in the Williams-Tukey's balanced model formula (Williams, 1962; Tukey, 1955). The resulting interval is

$$\left[\frac{(n_k s_y^2 - M \bar{y}_k F_{1, (k-1), (N-k)})}{n_k F_{1, (k-1), \alpha/2}}, \frac{(n_k s_y^2 - M \bar{y}_k F_{1, (k-1), (N-k)})}{n_k F_{1, (k-1), 1-\alpha/2}} \right] \quad (4.42)$$

By replacing $F_{1, (k-1), \alpha/2}$ with $\frac{1}{\chi^2_{1, (k-1)}}$ and $F_{1, (k-1), 1-\alpha/2}$ with $\frac{1}{\chi^2_{1, (k-1)}}$, formula (4.42) is equivalent to the formula given by Thomas and Holguin (1978, p. 386). The lower and upper bounds of (4.42) are defined to be zero if $\frac{n_k s_y^2}{M \bar{y}_k} < F_{1, (k-1), (N-k)}$ and $\frac{n_k s_y^2}{M \bar{y}_k} < F_{1, (k-1), (N-k)}$ respectively since $\sigma_1^2 \geq 0$. Thomas and Holguin mentioned that, in situations where an approximation of the W -random variable works well, this formula has a good coverage probability. We shall refer to formula (4.42) as the TH confidence interval.

Barlett-Rickman's [195] Confidence Interval

An approximate interval is said to be conservative if its coverage probability is larger than the nominal value, and liberal if it is smaller than the nominal value. In an unreplicated balanced design where $q < 1/20$, the Thomas-Balaguant confidence interval on σ_e^2 can be liberal (Barlett and Graybill 1993, p. 72). The approximate $(1 - \alpha)100\%$ confidence interval on $\sigma_e^2 + \frac{1}{q} \cdot \sigma_e^2$ based on Thomas-Balaguant's approximation can also be liberal, and Barlett, Mackay and Graybill's (1993) confidence interval on $\frac{\sigma_e^2}{q}$ is conservative since it guarantees a coverage probability at least as large as $1 - \alpha$. Combining these two intervals using Wilcoxon' (1962) strategy for constructing a confidence interval on σ_e^2 in the balanced design, Barlett and Rickman (1956) introduced a confidence interval on σ_e^2 for the unbalanced one-way random model. It is given by

$$\left[\frac{\sigma_e^2 \hat{L}_m}{F_{1-q}(k-1, n)(1 + n_0 \hat{L}_m)}, \frac{\sigma_e^2 \hat{U}_m}{F_{1-q}(k-1, n)(1 + n_0 \hat{U}_m)} \right], \quad (3.47)$$

where

$$\begin{aligned} \hat{L}_m &= \frac{\hat{S}_e^2}{F_{1-q}(k-1, N-k)(M\hat{S}_e)} = \frac{1}{n_{0L}}, \\ \hat{U}_m &= \frac{\hat{S}_e^2}{F_{1-q}(k-1, N-k)(M\hat{S}_e)} = \frac{1}{n_{0U}}, \\ n_{0L} &= \min\{n_0, n_0 + n_{0L}\}, \\ n_{0U} &= \min\{n_0, n_0 + n_{0U}\} \end{aligned}$$

and n_0 and $\hat{S}_e^2$ are defined as in (3.41). Again, the lower and upper bounds of (3.47) are all positive when $\hat{L}_m > 0$ and $\hat{U}_m > 0$, respectively. We shall refer to bounds (3.47) as the RE confidence interval on σ_e^2 .

Modified Thomas Mean (MTM) Confidence Interval

Consider the matrix $\mathbf{K} = \text{diag}(\frac{1}{k_1}, \frac{1}{k_2}, \dots, \frac{1}{k_k})$. The matrix $\mathbf{K}(\mathbf{I}_k - \frac{1}{k}\mathbf{J}_k)$ has $k-1$ positive eigenvalues, namely $k_1, k_2, \dots, k_{k-1}$. The average of these eigenvalues is the same as the reciprocal of the harmonic mean of the k_1 values, i.e. $\frac{1}{k}(\mathbf{I}_k - \frac{1}{k}\mathbf{J}_k)^{-1} \mathbf{I}_k = \frac{1}{k} \sum_{i=1}^k \frac{1}{k_i} = \frac{1}{\bar{k}}$. Klee (1999) proposed replacing n_k in the TM confidence interval (3.46) with the value $n_k = \frac{1}{\bar{k}} \sum_{i=1}^k k_i$, where $k_{(1)}$ and $k_{(k-1)}$ are the minimum and maximum of k 's, respectively. The resulting interval is c_k^* as

$$\left[\frac{n_k S_k^2 - MS_k F_{1-\beta}(k-1, N-k)}{n_k F_{1-\beta}(k-1, N-k)}, \frac{n_k S_k^2 - MS_k F_{1-\beta}(k-1, N-k)}{n_k F_{1-\beta}(k-1, N-k)} \right] \quad (3.46')$$

The lower and upper bounds of this interval are set to zero if $\frac{n_k S_k^2}{2MS_k} < F_{1-\beta}(k-1, N-k)$ and $\frac{n_k S_k^2}{2MS_k} < F_{1-\beta}(k-1, N-k)$, respectively. The choice of n_k is suggested to improve the approximation of W in Thomas and Holquist (1973) with the divergent matrix $\mathbf{V}_{k-1}^{-1}$. We still refer to formula (3.46) as the MTM confidence interval.

4.4.2. Modeling the coverage probability

The coverage probability of a confidence interval is c_k^* with a nominal value $1-\alpha$ depends not only on the variance components σ_1^2 and σ_2^2 , but also on the design, $\mathcal{D}$, that is the set of n_k values for model (3.1). Therefore, it is of interest to study the effects of these factors on the coverage probability of a confidence interval. The effects of variance components on the coverage probability have been studied using a variety of methods by several researchers as we have seen in Section 4.4.1. In all of these studies, only a few designs that were moderately or extremely unbalanced were used. These designs did not cover a wide range of unbalancedness. Moreover, it was hard to detect the combined effects of variance components and design on the coverage probability.

Table 4.18: Number of designs generated for each combination of N , ϕ , and p

N	50			100			500		
ϕ	0.20	0.40	0.60	0.20	0.40	0.60	0.20	0.40	0.60
p 0.1	5	7	10	7	11	16	9	15	20
p 0.5	5	7	10	7	11	16	9	15	20
p 0.9	5	7	10	7	11	16	9	15	20

As was seen earlier in Chapter 4, a design D for the model (2.1) can be generated by specifying the total number of observations, N , and the degree of imbalance, ϕ , for a fixed number of treatments, k , using Khan's [199] method. Such a design is not necessarily balanced. In this section, we provide an empirical model to represent the coverage probability of a confidence interval in α_0^2 in terms of the parameters N , ϕ , and the variance component through $p = \sigma_0^2/\sigma_1^2$. From this empirical relationship we can acquire a deeper insight into the effects of these parameters on the coverage probability of a confidence interval.

To demonstrate the development of such an empirical model, let us consider, as an example, that $k = 7$. Thus, $D = \{y_0, y_1, \dots, y_6\}$. We consider a 2^3 factorial arrangement for N , ϕ , and p with the selected levels, $N = 50, 100, 500$, $\phi = 0.20, 0.40, 0.60$, and $p = 0.1, 0.5, 0.9$. For each arrangement of the triple $\{N, \phi, p\}$, designs are generated on the basis of Khan's [199] method. Table 4.18 lists the number of designs generated for each of the 27 combinations of the levels of N , ϕ , and p . A total of 800 designs were thus obtained. Some of these designs are listed in Table 4.19, along with the actual value of ϕ for a ϕ -generated design.

First confidence interval formulas, as described in Section 4.4.1, are used in this investigation. To obtain an approximate coverage probability of a confidence interval,

Table 6.13: Generalized designs for the two-way model with $k=7$ (a, denotes the actual value of $\hat{\theta}$ for a $\hat{\theta}$ -generalized design).

$\hat{\theta}$	$\hat{\theta}$	$\hat{\theta}$	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
10	0.10	0.1	1	1	10	10	1	1	1	0.1001
10	0.10	0.1	1	1	10	1	1	5	5	0.1001
10	0.10	0.1	10	4	3	3	1	3	3	0.1001
10	0.10	0.1	0	2	1	1	6	10	1	0.1001
10	0.10	0.1	4	10	1	1	6	1	1	0.1001
10	0.10	0.0	1	1	1	10	1	1	1	0.1001
10	0.10	0.0	1	4	1	1	10	4	4	0.1001
100	0.10	0.1	11	11	10	1	1	11	10	0.1001
100	0.10	0.1	10	10	1	1	6	4	11	0.1001
100	0.10	0.0	10	10	10	4	1	4	11	0.1001
100	0.10	0.0	10	10	11	1	1	11	11	0.1001
100	0.10	0.0	11	1	1	11	11	11	1	0.1001
100	0.10	0.0	1	11	1	7	11	11	1	0.1001
100	0.10	0.1	11	11	11	11	1	11	1	0.1001
100	0.10	1.0	71	61	61	71	11	11	61	0.1001
100	0.10	1.0	11	11	11	11	71	71	11	0.1001
100	0.10	1.0	61	71	61	61	11	61	61	0.1001
100	0.10	1.0	61	61	11	61	71	61	11	0.1001
100	0.10	1.0	61	61	61	71	61	61	61	0.1001
100	0.10	1.0	71	61	61	61	71	11	71	0.1001

for a given design, D , and a p -value, a simulation study is performed. In this study, $\alpha = 0.05$ is used. The tabulated F values then used in the simulation are follows:

$$\begin{aligned} F_{\text{tab}}(0, \infty) &= 0.0002 & F_{\text{tab}}(0, \infty) &= 0.0002 \\ F_{\text{tab}}(0.01, \infty) &= 0.0004 & F_{\text{tab}}(0.01, \infty) &= 1.0000 \\ F_{\text{tab}}(0.05, \infty) &= 0.0008 & F_{\text{tab}}(0.05, \infty) &= 1.0004 \\ F_{\text{tab}}(0.10, \infty) &= 1.0 & F_{\text{tab}}(0.10, \infty) &= 1.0 \\ F_{\text{tab}}(0.50, \infty) &= 0.0002 & F_{\text{tab}}(0.50, \infty) &= 0.0002 \\ F_{\text{tab}}(1.00, \infty) &= 0.0002 & F_{\text{tab}}(1.00, \infty) &= 0.0002 \\ F_{\text{tab}}(5.00, \infty) &= 0.0002 & F_{\text{tab}}(5.00, \infty) &= 0.0002 \end{aligned}$$

These values are obtained by extrapolating the F tabulated values from Durbin and Graybill (1967). An outline of the simulation is described as follows:

- (a) Without any loss of generality we can assume that $\mu = 0$ in model (3.1). We then have

$$y_{ij} = \alpha_i + \epsilon_{ij}, \quad i = 1, 2, \dots, k; \quad j = 1, 2, \dots, n_i. \quad (4.4)$$

Also, without loss of generality, we assume $\sigma_1^2 + \sigma_2^2 = 1$. Therefore, by recalling that $\mu = \sigma_1^2 \frac{\partial}{\partial \mu} \ln L(\mu) = N \bar{y}_{1..} \mu$ and $\sigma_2^2 = N \bar{y}_{2..} (1 - \mu)$,

- (b) For a given p -value and a given design, $D = (n_1, n_2, \dots, n_k)$, a random vector $\mathbf{y}$ is generated on the basis of model (4.4). For this simulation, a FORTRAN program is developed using Park and Miller's (1978) random number generator.

- (c) With a simulated data vector $\mathbf{y}$, two-sided 95% confidence intervals are computed on σ_1^2 using each of the four formulas for a confidence interval on σ_1^2 (or equivalently, μ), namely, MLR, TR, BL, and MLEF.

- (d) Steps (b) and (c) are repeated a sufficient number of times. An estimated coverage probability, $\hat{p}$, of a confidence interval on σ_1^2 is then obtained by calculating the proportion of the intervals that contain the p -value.

- (b) Repeat steps (b), (c), and (d) for different specification of a μ value and a design D .

A total of 32,000 simulations of y in step (b) were conducted for each method. Estimated coverage probabilities of the four methods, namely, *FWER*, *FW*, *Post- μ FWER*, and *post- μ FW* are given in Table 4.20. Some several designs are generated for each selection of N and k , values of μ_0 , i.e., MLR, TR, BE, and MMR, corresponding to the same triple (N, k, ρ) represent replicated response values with N , k , and ρ acting as control variables. The data in Table 4.20 can now be used to model p_0 against N , k , and ρ for each method.

Given the nature of the response p_0 , being a probability, it would be appropriate to use generalized linear models. For values of μ_0 , i.e., MLR, TR, BE, and MMR, we consider the transformation

$$w_i^{\mu} = \left[\frac{e^{\beta_i \mu}}{1 + e^{\beta_i \mu}} \right]^m, \quad m \geq 1, \quad i = \text{MLR, TR, BE, and MMR}, \quad (4.26)$$

which maps the interval $[0, 1]$ onto $[0, w_i^{\mu}]$. Some several regressions are available on p_0 and hence on w_i^{μ} , i.e., MLR, TR, BE, and MMR, for each triple (N, k, ρ) in Table 4.20: the mean W_0^{μ} and sample variance $s_{W_0^{\mu}}^2$ of w_i^{μ} for i^{th} triple, $i = 1, 2, \dots, 17$, can be computed. It denotes a possible linear relationship between the mean and constant of w_i^{μ} or between the mean and standard deviation of w_i^{μ} : a simple linear regression model without an intercept using W_0^{μ} and $s_{W_0^{\mu}}^2$ or W_0^{μ} and $s_{W_0^{\mu}}$ is fitted. On the basis of the results in Table 4.20, the coefficient of determination (R^2) for the regression models are follows. For the MLR method, the R^2 value based on using W_0^{μ} and $s_{W_0^{\mu}}^2$ is 0.60 for $m = 1$, 0.45 for $m = 2$ and 0.33 for $m = 3$, whereas using W_0^{μ} and $s_{W_0^{\mu}}$ results in an R^2 value of 0.60 for $m = 1$, 0.47 for $m = 2$ and 0.34 for $m = 3$. For the TR method, the R^2 value for W_0^{μ} and $s_{W_0^{\mu}}^2$ is 0.79 for $m = 1$, 0.45 for $m = 2$ and 0.35 for $m = 3$, whereas using W_0^{μ} and $s_{W_0^{\mu}}$ gives an R^2 value of 0.82 for $m = 1$, 0.54 for $m = 2$ and 0.35 for $m = 3$. For the BE method, the R^2 value for W_0^{μ} and $s_{W_0^{\mu}}^2$ is 0.58 for $m = 1$, 0.77 for $m = 2$ and 0.54 for $m = 3$, whereas using W_0^{μ} and $s_{W_0^{\mu}}$

Table 4.18 The estimated coverage probability of χ^2 by simulation using four methods for each of the generated designs in Table 4.16 (nominal confidence coefficient is 0.05)

n_1	n_2	n_3	n_4	n_5	n_6	n_7	r	P_{exact}	P_{TC}	P_{MC}	P_{EMC}
2	1	23	12	1	1	1	0.1	0.0491	0.0494	0.0493	0.0491
5	1	34	1	1	-5	5	0.1	0.0495	0.0495	0.0493	0.0495
26	4	3	3	1	3	3	0.1	0.0498	0.0493	0.0493	0.0492
9	2	1	1	4	33	1	0.1	0.0493	0.0494	0.0493	0.0494
4	33	1	1	9	1	1	0.1	0.0487	0.0495	0.0494	0.0494
3	9	3	34	2	2	1	0.9	0.0477	0.0500	0.0492	0.0500
1	4	1	1	33	4	9	0.9	0.0423	0.0500	0.0491	0.0513
								•			
								•			
								•			
15	19	33	1	1	11	30	0.9	0.0490	0.0479	0.0494	0.0493
39	39	9	5	4	4	11	0.9	0.0477	0.0515	0.0497	0.0519
35	39	34	4	1	4	17	0.9	0.0437	0.0440	0.0493	0.0427
34	39	19	9	1	13	15	0.9	0.0491	0.0513	0.0493	0.0503
15	9	1	31	11	17	1	0.9	0.0434	0.0444	0.0490	0.0519
9	33	1	7	34	19	9	0.9	0.0499	0.0474	0.0479	0.0499
19	12	11	30	9	13	1	0.9	0.0491	0.0497	0.0490	0.0503
								•			
								•			
								•			
79	91	45	74	75	10	49	0.9	0.0491	0.0493	0.0493	0.0493
53	39	10	59	73	39	34	0.9	0.0497	0.0491	0.0494	0.0493
64	75	55	64	37	59	67	0.9	0.0491	0.0499	0.0490	0.0493
47	89	14	97	74	41	13	0.9	0.0495	0.0509	0.0490	0.0500
46	61	41	73	57	54	45	0.9	0.0495	0.0509	0.0490	0.0503
77	43	59	99	79	12	75	0.9	0.0471	0.0502	0.0490	0.0494

produces an R^2 value of 0.86 for $m = 1$, 0.90 for $m = 2$ and 0.96 for $m = 3$. Finally for the MRM method, the R^2 value corresponding to w_1^2 and w_2^2 is 0.86 for $m = 1$, 0.92 for $m = 2$ and 0.74 for $m = 3$, whereas for w_1^2 and w_3^2 the R^2 value is 0.93 for $m = 1$, 0.95 for $m = 2$ and 0.95 for $m = 3$. This indicates that there exists a strong linear relationship between the mean and standard deviation of w_i^2 for i -MLL, TH, TL, and MRM. With this transformation, a generalized linear model can be fitted using a gamma distribution for w_i^2 , i -MLL, TH, TL, and MRM. To relate the linear predictor, ξ , to the mean of w_i^2 , $\mu_{w_i^2}$, a logarithmic link function is chosen. This link function was preferred over the canonical link function for the gamma distribution because it produced a more flexible estimate of the coverage probability. Hence, $\xi = \log(\mu_{w_i^2})$, i -MLL, TH, TL, and MRM. Since $0 < \theta \leq 1$ and $0 < \rho < 1$, we use the variable $x = \frac{\theta}{1-\theta}$ instead of using θ itself in order to reduce the range of θ from still to 1. Hence, the region of interest determined by the selected values of θ , ρ and ρ is Table 4.4 is

$$C = \{(x, \rho, x) \mid \frac{50}{100} \leq x \leq \frac{100}{100}, 0.20 \leq \rho \leq 0.80, 0.1 \leq x \leq 0.9\} \quad (4.14)$$

The linear predictor considered is $\xi = \mathbf{f}(\mathbf{x})\boldsymbol{\beta}$ where $\mathbf{x} = (x, \rho, x)^T$, $\boldsymbol{\beta}$ is a vector of unknown parameters, and $\mathbf{f}(\mathbf{x})$ is a vector function of $\mathbf{x}$. Among several model forms of the linear predictor, a model having all two-factor interactions and a three-factor interaction is chosen based on its model deviance and model chi-squared statistic. The GENMOD procedure in SAS (1997) is used to fit the model. The fitted model which represents the linear predictor, ξ , is obtained as follows.

For the MLL method

$$\begin{aligned} \mathbf{f}(\mathbf{x})\boldsymbol{\beta} = & -8.4726 - 5.7441x + 5.8594x - 5.4084\rho + 5.6551\rho^2 \\ & + 3.4545\rho x + 4.7348\rho x = 3.2016\rho x \end{aligned} \quad (4.15)$$

For the TH method,

$$\begin{aligned} \hat{f}(x)\hat{\sigma}^2 = & -8.3993 + 1.2994x + 2.6602y + 2.7833z - 1.3640xy \\ & - 1.4050yz - 3.5418xz + 2.9580xyz \end{aligned} \quad (8.32)$$

For the BE method,

$$\begin{aligned} \hat{f}(x)\hat{\sigma}^2 = & 17.1400 - 4.4200x - 4.9013y - 5.7764z + 2.7044xy \\ & + 4.8264yz + 7.7711xz - 3.6413xyz \end{aligned} \quad (8.33)$$

For the MEM method,

$$\begin{aligned} \hat{f}(x)\hat{\sigma}^2 = & -3.9477 + 1.9086x + 3.1655y + 4.8430z - 2.1626xy \\ & - 2.7507yz - 4.2847xz + 3.0460xyz \end{aligned} \quad (8.34)$$

Tables 4.21, 4.22, 4.23, and 4.24 show the standard errors, and the associated P -value, of the corresponding test statistic for each of four methods (OLS, TH, BE, and MEM) respectively. Note that the same model form is used to fit the data in Table 4.16, so that we can compare the behavior of these methods in terms of their coverage probability as σ_0^2 , which is the topic of the next section.

Let $\mu_0(x)$ denote a predicted value of μ_0 (i.e. OLS, TH, BE, and MEM) at a given x inside the region of experimentation $\mathcal{U}$ in (4.11). Using models (8.30) through (8.34), $\mu_0(x)$ can be obtained, at a given point x , by

$$\hat{\mu}(x) = \frac{1}{1 + x'(\hat{F}^2)^{-1}x}, \quad (8.35)$$

for each method of (i.e. OLS, TH, BE, and MEM) respectively. Note that there are several replications of $\hat{\mu}_0$ (i.e. OLS, TH, BE, and MEM) at each triple (x, y, z) . Therefore, the maximum difference, as well as the average difference, between the estimated coverage probabilities as σ_0^2 obtained by simulation with method x , which is $\hat{\mu}_0$ versus the replication and the corresponding predicted coverage probability obtained by model (8.35), which is $\mu_0(x)$, (i.e. OLS, TH, BE, and MEM) at $x = (x, y, z)$

are calculated. These values will determine how good the fitted model is to represent an estimated coverage probability on σ_0^2 by simulation, and hopefully the true coverage probability on σ_0^2 . The predicted coverage probability by the model, $\hat{p}_0(\mathbf{x})$ and its maximum as well as average differences from p_0 at $\mathbf{x} = (x, \hat{\sigma}_0^2, x)$ are shown in Tables 4.26–4.28, 4.29, and 4.30 for each method of t-MLE, TE, BE, and MEM, respectively.

We note that values of the maximum difference range from -0.0048 to 0.0047 for t-MLE, from -0.0112 to 0.0097 for t-TE, from -0.0096 to 0.0176 for t-BE, and from -0.0094 to 0.0077 for t-MEM, whereas values of the average difference range from -0.0022 to 0.0046 for t-MLE, from -0.0076 to 0.0033 for t-TE, from -0.0045 to 0.0030 for t-BE, and from -0.0070 to 0.0032 for t-MEM. These values and the goodness of fit test statistics in Tables 4.31 through 4.34 indicate that model (4.54) with $\hat{\sigma}_0^2$, t-MLE, TE, BE, and MEM, given by (4.32) through (4.34) represents the data in Table 4.26 very well.

To see the effect of the explanatory variable $\mathbf{x}$ on the coverage probability on σ_0^2 , contour plots of $\hat{p}_0(\mathbf{x})$ for each method of t-MLE, TE, BE, and MEM, are made. Contour plots for t-MLE are shown in Figures 4.4, 4.5, and 4.6. Those for t-TE are in Figures 4.11, 4.12, and 4.13. Figures 4.14, 4.15, and 4.16 are for t-BE, whereas Figures 4.17, 4.18, and 4.19 are for t-MEM.

For the MLE method, the predicted coverage probability approaches the nominal value, namely 0.95, namely as $\hat{\sigma}$ increases when $\hat{\sigma}$ is fixed [see Figure 4.4]. For large N , $\hat{\sigma}$ is more influential than μ [see Figure 4.5(a)]. When $\hat{\sigma}$ is fixed, a higher level of coverage probability is obtained by decreasing both N and μ simultaneously [see Figure 4.6]. If a design is extremely unbalanced ($\mu = 0.05$), the MLE method produces a smaller coverage probability than the nominal value, regardless of the size of N and μ [see Figure 4.5(a)]. However, for a nearly balanced design ($\mu = 0.50$), a desirable level of coverage probability can be obtained with small N and μ values [see

Table 4.10: The standard errors of parameter estimates and goodness-of-fit statistics for model (1.52)

Model Deviance = 505.8758, $df = 392$ (value/ $df = 1.3049$)					
Model χ^2 value = 571.9687, $df = 392$ (value/ $df = 1.4590$)					
variable	parameter	estimate	std. err.	χ^2 stat.	P value
	β_1	1.4739	0.1471	10.04 399.0	0.0001
α	β_1	0.7643	0.2938	6.69 0.122	0.0001
ϕ	β_0	0.1864	0.2093	1.4993	0.0014
ρ	β_0	-0.8244	0.2653	9.69 5.45	0.0001
$\omega\phi$	β_{10}	3.6623	0.2794	16.6934	0.0001
$\omega\rho$	β_{10}	1.4840	0.3227	14.3335	0.0001
$\phi\rho$	β_{20}	4.7558	0.3058	22.1 31.3	0.0001
$\omega\phi\rho$	β_{20}	3.2854	0.3823	19.6233	0.0001

Table 4.11: The standard errors of parameter estimates and goodness-of-fit statistics for model (3.52)

Model Deviance = 320.8758, $df = 392$ (value/ $df = 1.8379$)					
Model χ^2 value = 368.1113, $df = 392$ (value/ $df = 0.9390$)					
variable	parameter	estimate	std. err.	χ^2 stat.	P value
	β_1	0.1829	0.1287	10.69 39.0	0.0001
α	β_1	1.3281	0.1713	11.1447	0.0001
ϕ	β_0	3.6995	0.1433	266.7266	0.0001
ρ	β_0	2.6629	0.2513	56.1 13.6	0.0001
$\omega\phi$	β_{10}	1.8835	0.2563	37.4267	0.0001
$\omega\rho$	β_{10}	-1.1825	0.2645	49.6153	0.0001
$\phi\rho$	β_{20}	3.6113	0.2713	147.1953	0.0001
$\omega\phi\rho$	β_{20}	1.9760	0.2839	29.4917	0.0001

Table 4.23. The standard errors of parameter estimates and goodness-of-fit statistics for model (3.56)

Scaled Sum-squares = 304.8793, $df = 392$ (value/ $df = 1.033$)					
Scaled χ^2 -value = 393.6327, $df = 392$ (value/ $df = 1.129$)					
variable	parameter	estimate	std. err.	χ^2 stat.	P-value
x	β_0	17.3808	0.2296	1338.3493	0.0001
	β_1	-4.4939	0.2990	211.6399	0.0001
y	β_0	-4.7082	0.4092	409.1791	0.0001
x	β_0	0.7790	0.2664	861.6637	0.0001
xy	β_{xy}	3.8963	0.4063	113.6768	0.0001
xy	β_{xy}	4.6497	0.4843	112.7143	0.0001
xy	β_{xy}	7.7741	0.4580	290.4763	0.0001
xy	β_{xyz}	1.9483	0.3754	26.3318	0.0001

Table 4.24. The standard errors of parameter estimates and goodness-of-fit statistics for model (3.57)

Scaled Sum-squares = 305.6528, $df = 393$ (value/ $df = 1.008$)					
Scaled χ^2 -value = 375.7394, $df = 393$ (value/ $df = 0.958$)					
variable	parameter	estimate	std. err.	χ^2 stat.	P-value
x	β_0	1.1672	0.1697	1293.1273	0.0001
	β_1	1.1366	0.2266	72.1777	0.0001
y	β_0	1.1833	0.3636	101.1988	0.0001
x	β_0	4.0409	0.3917	104.1688	0.0001
xy	β_{xy}	-1.1809	0.2936	16.1974	0.0001
xy	β_{xy}	-1.7697	0.2756	16.4103	0.0001
xy	β_{xy}	-4.1187	0.2162	146.6492	0.0001
xy	β_{xyz}	1.1440	0.4753	37.1867	0.0001

Table 4.15: The predicted coverage probability on σ_0^2 using model (4.10), $\mu(\theta)$ and its maximum and average difference with the estimated coverage probabilities using simulation $\hat{p}$ for given values of $n = \lfloor n_0, n \rfloor$ when the ML method is used

N	$n = \frac{n_0}{\rho_0}$	δ	ρ	$\hat{p}(\sigma)$	maximum diff	average diff
50	5.1111	0.10	0.1	0.93020	-0.00000	-0.00000
			0.5	0.95040	0.00000	-0.01430
			0.9	0.99041	-0.00000	-0.01617
	0.15	0.1	0.10000	0.93040	-0.00000	-0.00000
			0.5	0.93943	0.00000	0.00000
			0.9	0.99000	0.00000	0.00000
	0.55	0.1	0.10000	0.93040	-0.00000	-0.00000
			0.5	0.93943	0.00000	0.00000
			0.9	0.99000	0.00000	0.00000
150	0.3333	0.30	0.1	0.93177	-0.00000	0.00000
			0.5	0.93187	0.00000	0.00000
			0.9	0.99000	-0.00000	-0.00000
	0.45	0.1	0.10000	0.93187	-0.00000	0.00000
			0.5	0.93187	0.00000	0.00000
			0.9	0.99000	-0.00000	-0.00000
	0.75	0.1	0.10000	0.93187	-0.00000	0.00000
			0.5	0.93187	0.00000	0.00000
			0.9	0.99000	-0.00000	-0.00000
400	0.2500	0.30	0.1	0.93017	0.00000	0.00000
			0.5	0.93017	0.00000	0.00000
			0.9	0.99000	0.00000	-0.00000
	0.55	0.1	0.10000	0.93017	0.00000	0.00000
			0.5	0.93017	0.00000	0.00000
			0.9	0.99000	0.00000	0.00000
	0.95	0.1	0.10000	0.93017	0.00000	0.00000
			0.5	0.93017	0.00000	0.00000
			0.9	0.99000	0.00000	0.00000

Table 4.26: The predicted average probability on $\mathcal{P}_0^2$ using model (4.44), $\hat{p}(x)$ and its maximum and average difference with the estimated average probability using simulation $\hat{p}$ for given values of $n = (n, d, \rho)$ when the TD method is used

N	$n = \frac{N}{p_0}$	d	ρ	$\hat{p}(x)$	maximum diff	average diff
50	8 (1/1.1)	0.40	-0.1	0.99964	-0.00033	0.00000
			-0.0	0.99991	-0.00009	-0.00000
			0.1	0.99999	0.00001	0.00000
		0.45	-0.1	0.99909	-0.00091	-0.00043
			-0.0	0.99821	-0.00179	-0.00099
			0.1	0.99779	-0.00221	-0.00106
	0.50	0.1	-0.1	0.99480	-0.00520	-0.00263
			-0.0	0.99000	-0.00999	-0.00500
			0.1	0.99000	-0.00999	-0.00500
		0.15	-0.1	0.99000	-0.00999	-0.00500
			-0.0	0.99000	-0.00999	-0.00500
			0.1	0.99000	-0.00999	-0.00500
100	8 (0.020)	0.30	-0.1	0.99977	-0.00023	0.00000
			-0.0	0.99999	-0.00001	-0.00000
			0.1	0.99999	0.00001	0.00000
		0.40	-0.1	0.99983	-0.00017	0.00000
			-0.0	0.99997	-0.00003	-0.00000
			0.0	0.99999	0.00001	0.00000
	0.30	0.1	-0.1	0.99999	-0.00001	-0.00000
			-0.0	0.99999	0.00001	0.00000
			0.0	0.99999	0.00001	0.00000
		0.15	-0.1	0.99999	-0.00001	-0.00000
			-0.0	0.99999	0.00001	0.00000
			0.0	0.99999	0.00001	0.00000
500	1 (0.002)	0.30	-0.1	0.99979	-0.00021	0.00000
			-0.0	0.99979	-0.00021	-0.00000
			0.0	0.99999	0.00001	0.00000
		0.40	-0.1	0.99999	-0.00001	0.00000
			-0.0	0.99977	-0.00023	-0.00000
			0.0	0.99979	-0.00021	0.00000
	0.30	0.1	-0.1	0.99979	-0.00021	0.00000
			-0.0	0.99979	-0.00021	-0.00000
			0.0	0.99979	-0.00021	0.00000
		0.15	-0.1	0.99979	-0.00021	0.00000
			-0.0	0.99979	-0.00021	-0.00000
			0.0	0.99979	-0.00021	0.00000

Table 4.17: The predicted average probability on $\mathcal{A}_1^0$ using model (4.14), $\hat{p}(x)$, and its maximum and average difference with the estimated average probabilities using resolution, $\hat{p}$, for given values of $n = \{n_1, n_2, n\}$ when the MC method is used

N	$n = \frac{n_1}{n_2}$	δ	ρ	$\hat{p}(x)$	maximum diff.	average diff.
50	0.15.1.15	0.05	0.1	0.94088	0.00002	0.00002
			0.5	0.94985	0.00005	0.00003
			0.9	0.94943	0.00005	-0.00005
		0.15	0.1	0.94901	0.00008	0.00005
			0.5	0.95326	0.00005	0.00003
			0.9	0.95733	-0.00003	0.00001
		0.55	0.1	0.95951	-0.00002	-0.00005
			0.5	0.95956	0.00005	0.00003
			0.9	0.95975	-0.00002	-0.00005
100	0.30.0.70	0.35	0.1	0.95058	0.00008	0.00003
			0.5	0.95186	0.00005	0.00003
			0.9	0.95637	-0.00002	-0.00005
		0.45	0.1	0.95457	0.00004	0.00003
			0.5	0.95456	0.00005	0.00004
			0.9	0.95978	-0.00001	-0.00005
		0.55	0.1	0.95970	0.00008	0.00005
			0.5	0.95681	0.00005	0.00003
			0.9	0.95957	0.00002	0.00007
150	1.00.0.00	0.99	0.1	0.95750	0.00003	0.00005
			0.5	0.95875	0.00004	0.00005
			0.9	0.95907	-0.00002	-0.00004
		0.95	0.1	0.95900	-0.00005	-0.00005
			0.5	0.95926	0.00003	0.00003
			0.9	0.95955	-0.00003	-0.00005
		0.93	0.1	0.95923	0.00005	0.00003
			0.5	0.95955	0.00005	0.00004
			0.9	0.95950	0.00003	0.00005

Table 4.26: The predicted average probability on π_1^0 using model (4.51), $p(x)$, and its maximum and average difference with the associated average probabilities using simulation (μ), for given values of $\theta = [a, b, c]$ when the MCMC method is used

N	$\pi = \frac{1}{\sqrt{2}}$	θ	ρ	$p(x)$	maximum diff.	average diff.
50	0.1113	0.40	0.1	0.00007	-0.00003	-0.00047
			0.5	0.00011	-0.00004	0.00044
			0.9	0.00009	0.00004	0.00004
		0.60	0.1	0.00000	0.00000	0.00000
			0.5	0.00004	0.00001	0.00001
			0.9	0.00009	0.00000	0.00003
		0.80	0.1	0.00000	-0.00004	-0.00000
			0.5	0.00004	-0.00000	-0.00001
			0.9	0.00009	0.00000	0.00000
100	0.33333	0.30	0.1	0.00001	0.00000	-0.00000
			0.5	0.00003	-0.00001	-0.00004
			0.9	0.00000	0.00000	0.00000
		0.60	0.1	0.00004	0.00000	0.00001
			0.5	0.00004	-0.00000	0.00000
			0.9	0.00009	0.00000	0.00001
		0.90	0.1	0.00000	-0.00001	-0.00000
			0.5	0.00000	0.00000	0.00000
			0.9	0.00000	-0.00000	0.00000
400	1.00000	0.30	0.1	0.00000	0.00000	0.00000
			0.5	0.00000	-0.00000	-0.00000
			0.9	0.00000	0.00000	0.00000
		0.60	0.1	0.00000	0.00000	0.00000
			0.5	0.00000	-0.00000	-0.00000
			0.9	0.00000	0.00000	0.00000
		0.90	0.1	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000

Figure 4.10(c)). When p is small, by increasing ϕ or decreasing N we can get a good coverage probability [see Figure 4.10(b)]. However, even when a design is being more unbalanced, a desirable level of coverage probability can be obtained by decreasing the size of N . More specifically, when p is as small as 0.1, to get at least 0.94 coverage probability on σ_0^2 , we need a design having $\phi = 0.65$ when $n = 1.5$ (i.e., $N = 400$). If the design is unbalanced with $\phi = 0.55$, it can provide at least 0.94 coverage probability on σ_0^2 when $n = 0.2$ (i.e., $N = 60$). When p is large, changing N is not as effective as increasing ϕ [see Figure 4.10(c)]. It is interesting to note that even when p is as large as 0.9, the MLR method gives fairly high coverage probability on σ_0^2 , which is 0.94, for a nearly balanced design ($\phi = 0.5$) regardless of the size of N . This shows that the use of the MLR formula provides a good coverage probability on σ_0^2 not only with small unbalanced p but also with large values of p as long as a design is nearly balanced. However, across the range of the parameter space we considered, the MLR method yields biased confidence intervals, particularly when ϕ is small and p is large.

For the TR method, the coverage probability approaches the nominal value as ϕ and p values are increased together when N is fixed [see Figure 4.11]. This holds across all N values considered. For a given ϕ value, the coverage probability increases as p increases, however, increasing N has a little effect [see Figure 4.12]. For a nearly balanced design ($\phi = 0.5$), the TR formula produces a fairly good coverage probability regardless of the size of N and p [see Figure 4.11(c)]. In Figure 4.11(c), only one contour line is drawn since the fitted model (3.16) with the TR formula produces coverage probabilities close to 0.94 when $\phi = 0.5$ for all the values of N and p considered. When the p values are small to moderate, the coverage probability increases as ϕ increases [see Figure 4.11(a,b)]. From Figure 4.12(a), it is interesting to note that even when p is small, we can have a good coverage probability if a design is nearly balanced ($\phi = 0.5$) regardless of the size of N , which implies that the

TE method is useful for $p \leq 0.33$ as long as the design is nearly balanced. On the contrary, with a large p value, the coverage probability tends to be overestimated for small values of N and d [see Figure 4.13(a)]. An approximate confidence interval with the TE method tends to be liberal when both d and p values are small, while it tends to be slightly conservative for a large p value.

For the BE method, we can obtain the nominal value of coverage probability by increasing p when either N or d is fixed [see Figures 4.14 and 4.15], and by increasing N and d together when p is fixed [see Figure 4.16]. However, the BE method generally produces from moderate to highly conservative confidence intervals except when p is large [see Figure 4.13(a)].

The behavior of the MEM confidence interval is very similar to that of TE's [see Figures 4.17, 4.18, and 4.19].

In summary, the MLE method is good for large values of d with small p values, however, it produces a highly liberal confidence interval when d is small, or p is large unless d is large. Both TE and MEM methods generally produce a good coverage probability unless both d and p are small and N is small to moderate. In this case, however, MLE is better than TE and MEM. The BE method is good only when p is large, or both N and d are large if p is small.

4.4.4 *Comparison of methods in terms of certain conditions*

The MLE procedure works well if d_1^2 is small. However, if d_1^2 is large, this procedure may result in very liberal intervals (Burdick and Graybill, 1968, p. 70). Although Burdick and Hawkins (1989) indicated that this procedure also yields liberal intervals in extremely unbalanced designs, the combined effect of the variance components and the design on this procedure has not yet been studied. Thomas and Hothorn (1974, p. 100) stated that "they would not expect such methods to give much better results, at least for the designs considered", that is, better results than their formula gave,

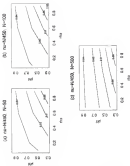


Figure 4.4: Contour plots of the predicted coverage probability in c_0^0 with model (4) (C) for each level of $\alpha = \frac{0.05}{m}$ using the Modified Large Sample (MLS) method

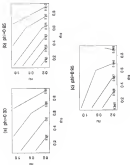


Figure 4.6: Cusiness plots of the perturbed coverage probability on σ_0^2 with model [4.14] for each level of δ ($\rho=\frac{1}{2}$) using the Modified Large Sample (MLS) method

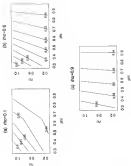


Figure 4.16: Coverage plots of the predicted coverage probability on ρ^2 with model (4.12) for each level of ρ ($\rho = \frac{\rho_0}{\rho_0 + 1}$) using the Modified Large Sample (MLS) method

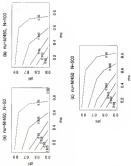


Figure 4.10: Contour plots of the posterior coverage probability on x_1^2 with model (4.14) for each level of $r = \frac{\sigma_1^2}{\sigma_2^2}$ using the Thomas-Balakrishnan's (TB) method

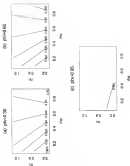


Figure 4.12. Contour plots of the predicted coverage probability on α_0^2 with model (4.12) for each level of α ($\alpha = \frac{\alpha_0}{\sqrt{2}}$) using the Thomas-Hinkley's (TH) method

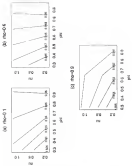


Figure 4.12: Contour plots of the pre-tested coverage probability on v_0^2 with model (4.52) for each level of ρ [$\rho \in \{0.1, 0.5, 0.9\}$] using the Thomas-Balakrishnan's (TB) method

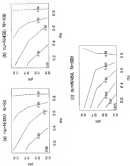


Figure 4.14: Contour plots of the predicted average percentage on α_1^2 with model (154) for mark level of $\alpha = \frac{1}{100}$ using the Bootstrap-Eulerian + (BE) method

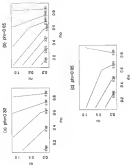


Figure 8.15 Cluster plots of the predicted average probability vs. r_0^2 with model (1.34) for each level of α ($r_0 = \frac{\sqrt{2}}{2}$) using the Bootstrap-Edwards's (BE) method

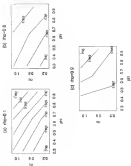


Figure 4.10: Contour plots of the predicted coverage probability on c_0^* with model (4.14) for the mark level of $p \in \{0.05, 0.1, 0.25\}$ using the Markov-Blochmann's (MB) method

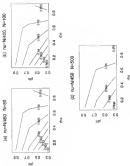


Figure 4.17: Contour plots of the predicted average probability on x_1^0 with model (4.32) for mark level of $\alpha = \frac{1}{10}$ using the Modified Bayesian Network (MBN) method

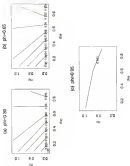


Figure 4.19: Contour plots of the predicted coverage probability on x^0 with model (4.30) for mark level of α ($\alpha = \frac{0.05}{2}$) using the Modified Bernstein Method (mBM) method

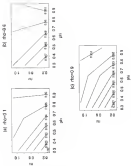


Figure 4.15: Contour plots of the predicted coverage probability on ρ^2 with model (4.17) for each level of α ($\alpha = \frac{\beta}{100}$) using the Modified Pearson Mean (mPM) method

since the approximation of the H -random variable with the chi-squared distribution worked very well for most non-ray subbalanced designs they had considered. Bartlett and Holman (1956) showed, using simulation study, that their procedure provided a higher coverage probability as σ_0^2 than Thomas-Holquist's procedure across the experimental situations they considered. They indicated that their procedure could always be recommended over the Thomas-Holquist's method (Bartlett and Holman, 1956, p. 311). They showed that their method had a coverage probability that was generally at least as large as the nominal value. It was also noted that, although normal (1- α) was more conservative than Thomas-Holquist's, these two methods produced confidence intervals whose lengths did not differ by more than 1%. Fleiss (1981) showed that, by replacing σ_0 with σ_{0n} , the approximation associated with the H -random variable in Thomas and Holquist (1956) is adequate when σ_0^2 is large. It was also shown that, when σ_0^2 is small, the approximation with σ_{0n} was slightly better than that with σ_0 for upper tail probabilities, while the former was slightly worse than the latter for lower tail probabilities. He also compared the use of σ_0 versus that of σ_{0n} with regard to the power of the F test concerning σ_0^2 .

In this section, we compare these five different methods of constructing confidence interval as σ_0^2 using their estimated coverage probabilities obtained by simulation. Partial comparisons based on contour plots using predicted coverage probabilities obtained by the fitted models were already made in the previous section. Table 4.28 shows the ranges of simulated coverage probabilities resulting from the replicates for a given triple (N, d, α) . Note that the nominal confidence coefficient is 0.95. Using the normal approximation to the binomial, if the true confidence coefficient is 0.95, there is less than 2.5% chance that an estimated confidence coefficient based on 10,000 replications will be less than 0.902 ($=0.95-1.96\sqrt{\frac{0.95(0.05)}{10000}}$). Also, there is less than 1.4% chance that an estimated confidence coefficient will be more than

0.9545 ± 0.0047) $95\sqrt{\frac{2.58^2(0.0047)^2}{1000}}$. Values which are significantly below the stated level in each triple (N, δ, ρ) are indicated using a symbol *, and those with significantly above the stated level are indicated using a symbol †. The MLE procedure generally provides liberal confidence intervals, but is good when δ is large and ρ is small. It is also good compared to the other procedures when both δ and ρ are small and N is not large. The TH and MEM procedures behave similarly, and produce liberal intervals when ρ is small. When ρ is moderate to large, the TH procedure reaches the nominal value of coverage probability when N is small. However, the MEM procedure tends to produce a slightly higher coverage probability than the nominal value for the moderate to large values of ρ , especially when δ is large. The BE procedure provides too-conservative intervals when ρ is small. This procedure is good with moderate to large values of ρ if δ is not too small.

Freeman's chi-square goodness-of-fit statistic is calculated over 300 generated designs, namely,

$$\chi^2_i = \sum_{j=1}^{100} \frac{(p_{ij} - 0.95)^2}{0.95}, \quad i = \text{MLE, TH, BE, and MEM},$$

in order to see the agreement of the estimated coverage probability obtained by simulation, with the nominal value over the range of experimentation G' is (3.53).

The values of χ^2_i are $\chi^2_{\text{MLE}} = 0.5456$, $\chi^2_{\text{TH}} = 0.0004$, $\chi^2_{\text{BE}} = 0.0000$, and $\chi^2_{\text{MEM}} = 0.0004$. The χ^2_i statistics calculated just over the replicates for each triple (N, δ, ρ) are listed in Table 4.10.

The average value of the estimated coverage probability obtained by simulation within the replicates for each triple (N, δ, ρ) is supposed to represent the true coverage probability for the same triple. First of the average estimated coverage probability against β for each combination of δ and ρ values are given in Figure 4.20. In these plots, the dashed line (—) is for the MLE method, the dotted line (····) is for the TH method, the short dashes line (- -) is for the MEM method, and the long

Table 4.26. Ranges of simulated average probability obtained with normal confidence coefficient of 95%

$M + \nu$	MLE	TE	PE	MEM
10 0.00 0.1	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.000° - 0.000°	0.00° - 0.00°
0.0	0.000° - 0.000°	0.00° - 0.000°	0.00° - 0.000°	0.00° - 0.000°
10 0.00 0.1	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.000°	0.00° - 0.000°	0.00° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°
10 0.00 0.0	0.00° - 0.00°	0.00° - 0.000°	0.000° - 0.000°	0.00° - 0.000°
0.0	0.00° - 0.00°	0.00° - 0.000°	0.00° - 0.000°	0.00° - 0.000°
0.0	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°
100 0.00 0.1	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.000°	0.00° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°
100 0.00 0.1	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°	0.000° - 0.000°
0.0	0.000° - 0.000°	0.000° - 0.00°	0.00° - 0.000°	0.00° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.000°
100 0.00 0.1	0.00° - 0.00°	0.00° - 0.000°	0.000° - 0.000°	0.00° - 0.000°
0.0	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°
0.0	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°
100 0.00 0.1	0.000° - 0.000°	0.000° - 0.00°	0.000° - 0.000°	0.000° - 0.00°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.000°	0.00° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°
100 0.00 0.1	0.000° - 0.000°	0.000° - 0.00°	0.000° - 0.000°	0.000° - 0.00°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.000°	0.00° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°
100 0.00 0.1	0.000° - 0.000°	0.000° - 0.00°	0.000° - 0.000°	0.000° - 0.00°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.000°	0.00° - 0.000°
0.0	0.000° - 0.000°	0.00° - 0.00°	0.00° - 0.00°	0.00° - 0.00°

Table 4(B) Goodness-of-fit statistic values obtained over the replicates for a given sample (N, n, ρ)

N	$n = \frac{N}{10}$	ρ	r	WLS	TS	LR	LRadj
100	0.10000	0.50	0.1	0.00007	0.00170	0.00077	0.00081
			0.5	0.00000	0.00000	0.00070	0.00000
			0.9	0.00000	0.00000	0.00007	0.00000
		0.85	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
	0.80	0.1	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
		0.5	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
100	0.10000	0.50	0.1	0.00000	0.00171	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
		0.85	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
	0.80	0.1	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
		0.5	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
100	0.10000	0.50	0.1	0.00000	0.00170	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
		0.85	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
	0.80	0.1	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000
		0.5	0.1	0.00000	0.00000	0.00000	0.00000
			0.5	0.00000	0.00000	0.00000	0.00000
			0.9	0.00000	0.00000	0.00000	0.00000

broken line (---) is for the BE method. The MLE method produces closer coverage probability to the nominal level than other methods with small to moderate size of N when ρ is small. However, if N is large, the TH and MEM methods are better than the MLE method when ρ is small. For moderate to large values of ρ , the TH, BE, and MEM procedures provide good coverage probabilities regardless of the size of N and α . Figure 4.28 is for the plots of the average coverage probability against θ for each combination of N and ρ values. A better performance of the TH procedure than the MEM procedure can be seen when $N = 50$ and $\rho = 0.1$, or when $N = 100$ and $\rho = 0.1$, for all values of θ . In both cases, however, the MLE method produces better coverage probability than both TH and MEM. When $N = 100$ and $\rho = 0.1$, the MEM method is best, though it is slightly better than the TH method. The coverage probability of the MLE procedure increases toward the nominal value as θ increases regardless of the size of N and ρ , however, its performance is poor compared to the other procedures. For large values of ρ , no distinction can be made among the TH, BE, and MEM procedures. Finally, plots of the average coverage probability against ρ for given values of N and θ are given in Figure 4.29. The coverage probability of the MLE and BE decreases as ρ increases. The TH and MEM methods behave in a similar pattern. They perform better than the BE method in general. They also produce better coverage probability than the MLE method except for small ρ and small to moderate values of N regardless of θ .

Based on the above plots we can summarize the behavior of the four methods as follows: The MLE method is preferred over the other methods when ρ is small and N is small to moderate. Otherwise, the TH and MEM methods give better coverage probability, and the distinction between these two methods is small. The BE method is also as good as the TH and MEM, but only when ρ is large.

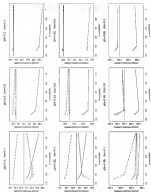


Figure 4.26 The average estimated coverage probability within the replicate subset n ($n = 500$) for each combination of levels of α and ρ . Lines represent the MLE method (—), the TH method (---), the MHT method (···), and the NI method (- · -).

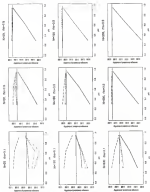


Figure 4.14. The average estimated coverage probability versus the significance level α for each combination of levels of ρ and n ($\rho = \frac{\sqrt{3}}{2}$). Lines represent the MLE method (—), the TR method (---), the MRM method (- · -), and the DE method (···).

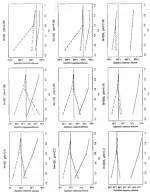


Figure 4.25: The average estimated coverage probability within the replicates against ρ for each combination of levels of α and λ ($\alpha = \frac{1}{100}$). Lines represent the MLE method (—) the TR method (---) the MRM method ($\cdots$) and the BE method (-.-).

4.4.3. Concluding remarks

The coverage probability of a confidence interval depends not only on the design used, but also on the values of the variance components. Several approximate confidence intervals on σ^2_{μ} for the one-way random model are available in the statistical literature. However, their performance in terms of the coverage probability as it relates to the variance components and the design has not been studied in the statistical literature.

In this study, we investigate the effect of imbalance as well as the effect of variance components on the coverage probability of four approximate confidence intervals. To do this, for each of the confidence intervals, an estimated coverage probability is obtained by simulation with a given design and a specified value of variance components. Thus, this probability is a function of the total number of observations, the degree of imbalance, and the ratio of the variance components for a fixed number of treatment levels. This probability is empirically modeled in terms of these variables using generalized linear models techniques for each method. The fitted model is used to observe the effect of these variables on the coverage probability. The four methods can then be compared on the basis of their respective models. Although no single method is considered best, the modified large sample interval is useful when p is small with nearly balanced design. It also performs better than the other intervals when p is small and σ^2 is small to moderate. The Thomas-Balaguer and the modified harmonic mean intervals perform well for intervals to large values of p regardless of the size of N and p . The Bonferroni-Kishman interval is also good for large values of p , however, it is too conservative if p is small.

CHAPTER 3 FURTHER KNOWLEDGE

Three extensions of the use of the QDOs in Kueh (MCT) were given in Chapter 2, namely, the application to the one-way, two-way cross-classification without interaction, and three-fold nested classification random models, the use of unbalanced data, and the consideration of maximum likelihood (ML) estimation. Further extensions are needed. For example, consideration of other methods of estimating variance components such as REML and improved ANOVA. The application of the methodology with general effects models are also of interest. Moreover, consideration of models with random effects having distributions other than the normal is of interest and can provide added flexibility to the method of comparing designs on the basis of QDOs or DQDOs.

In Chapter 4 we investigated the effect of imbalance on various quantities of interest. They were the probability of getting a superior ANOVA estimate of σ_e^2 , the power of the F test for σ_e^2 with homogeneity assumption concerning the error variances, the power of the same test, with heterogeneity assumption of error variances, and the coverage probability of a confidence interval on σ_e^2 . These investigations were carried out using an unbalanced one-way random model. An extension of the proposed methodology to higher-order models is of interest. Extending Kueh's (1977) method of generating designs for the one-way model to crossed classification models is straightforward. For the nested classification model, however, the extension can be more involved since different degrees of imbalance exist at different stages in the nesting relationship. Although designs can be generated separately for each stage

we need to have an overall measure of imbalance for all the stages in order to see the effect of imbalance of a design on a quantity of interest. The effect of imbalance on other quantities is also of interest. For example, the small sample behavior of the estimator can be examined by modeling the mean squared error of the estimator in terms of the variance components and the design. With a normal distribution model an investigation of the so-called "variance balancing effect" in Denham and Geyer (1979), which was briefly reviewed in Section 2.1, is of interest.

APPENDIX SPLUS CODE FOR GENERATING DESIGNS

Following is the SPLUS program to generate a design for the given value of the total number of observations, N , the degree of imbalance, d , and the correlation coefficient, $\rho = \frac{d}{N-d}$, with the many-way random model. The number of treatment group is fixed by $k = 3$ in the program. For other values of k , the function "initial.get.k" should be modified.

%%
 %%%%%%%%%%% for the following functions in Splus mode %%%%%%%%%%%
 %%%

%% program begins %%%

```

#
# Obtain Hadamard matrix for orthogonal transformation
#
initial.hadamard <- function(N){
  k <- matrix(0,k,k)
  for(i in 1:(N-1)){
    for(j in 1:(k-1))
      k[i,j] <- 1/(sqrt(k)+i+j+0.5)
  }
  Q(N,k) <- 1/(sqrt(k))
  Q(k,k) <- 1/(sqrt(k))
  for(i in 2:k){
    j <- i-1
    k[i,j] <- -1/(sqrt(k)+i+j)
  }
  Q
}

```

```

#
# Normal random numbers are used to generate random vectors
# uniformly distributed in a sphere
# For the algorithm of generating this random vectors,
# see Devroye (1982, pp. 228-232)
#
my random <- function(h,phi) {
  k1 <- h-1
  R <- matrix(0,k1,1)
  Z <- matrix(0,h,1)
  standard <- 0
  out <- matrix(0,h,h)
  n <- sqrt((1-phi)/2*log(1))
  flag <- 1
  while(flag==0 & standard<0.00001) {
    standard <- standard + 1
    R <- runif(k1,0,1)
    n <- sqrt(1-(R-R[1]))
    for(i in 1:k1) {
      R[i,2] <- runif(1,0,1)
    }
    Z[i,1] <- 1-sqrt(R[1])
    R <- subset(R,subset=2)
    Z <- Z[1:k1,2]
    out[i,1:R[i,1]] <-
      flag <- flag+1
    out <- Z
  }
}
out
}

#
# Calculates all possible combinations of Z's values in vector d (phi)
# This function is specifically for the test (2'W-SS) eqn(1)
#
test2 get d <- function(h2,h,br1) {
  nQs <- 0
  eqn["You chose an unknown function. This is just for h=2"]
  n <- 0
  d <- matrix(1-h,h2,h)
  for(j1 in 1:h2) {
    for(j2 in 1:h2) {
      for(j3 in 1:h2) {
        for(j4 in 1:h2)

```

```

        nextY is 1:20{
          nextZ is 1:20{
            nextY is 1:20{
              n ← n+1
              a[n,1] ← abs(sin(2*pi*x[j1],2*pi*x[j2],2*pi*x[j3],2*pi*x[j4],
                2*pi*x[j5]))
            }
            a[n,2] ← abs(sin(2*pi*x[j1],2*pi*x[j2],2*pi*x[j3],2*pi*x[j4],
              2*pi*x[j5],2*pi*x[j6],2*pi*x[j7]))
          }
        }
      }
    }
  }
}

#
# Currently, this function is good only for k=0
# For different k, you need to modify 'label.gen()' function
# suitable for that k value
# Input Information:
#   k = number of treatment groups
#   N = number of total sample size
#   phi = specified degree of imbalance
#   rho = signal/noise / (signal/noise + signal/no., 100)
#   max = number of designs used to generate given N, k, phi
# Output Results:
#   number of 'best' ones of
#   (N, phi, rho, a1,a2,a3,a4,a5, phi,best)
#   where
#   phi,best = actual phi(degree of imbalance) with a1, ... ,a5
#
my.optimal <- function(N,k,phi,rho,max){
  iter <- 0
  best <- matrix(00,max,k)
  res <- matrix(00,max,N+1)
  sz <- 2^k
  require <- F"1/(phi-1)
  for k <- matrix(1,k,1)
  flag <- 0
  while(flagmax < k + max/10000){
    iter <- iter + 1
    g[iter]
  }
}

```

```

    stop1 <- ~1
  else if (is.na(s1))
    stop1 <- matrix(-1, nrow=1, ncol=1)
  s <- as.matrix(s, p0)
  if (any(is.na(s)))
    stop("Generating uniform random matrix with all 1 > 0 is failed
      after 10000 iterations. Possible solutions are:
      - decrease the number of 'vars', or
      - increase 'B' and/or 'phi'")
  v <- matrix(0, k, 1)
  to <- matrix(0, k, 1)
  m <- matrix(0, k, 1)
  w <- B * s
  bb <- floor(bb)
  m <- sbinding(m)
  to <- sbinding(to, w)
  d <- cbind(p0, d[bb], k, to)
  if (any(is.na(d)))
    warning("failed to construct all possible combinations of vars")
  for (i in 1:k)
    for (j in 1:bb)
      if (any(d[j,]==0 & m[i,j]==0))
        count <- count+1
    }
  }
  d0 <- matrix(-1, nrow=k)
  posits <- 0
  for (i in 1:k)
    if (any(d[i,]==0 & m[i,]==0))
      posits <- posits+1
      d[posits,] <- d[i,]
  }
  }
  if (any(is.na(d)))
    warning("proper choice of good combinations is failed")
  m0 <- count
  any.m <- count/count * 1
  ncolumns <- sbinding(m-required)
  d.mvar <- sbinding(m, ncolumns)
  fixed.opts <- rowSums(mvar[, k+1])
  for (i in 1:count)
    if (is.na(d[i, k+1]) == fixed.opts[i])
      fixed <- d.mvar[i, ]
  }
  }

```

```

dropped <- findN(cdn=1)
optional <- !dropped
ordered <- sort(x=ordered)
reordered <- x[ordered]
phi.km <- P%Q%Qre[optional?"0 0"] row.km
flag <- flag*1
chr(flag,1) <- reordered
flag1 <- flag+1
writep=<nil>
write(1) <- write(0,phi,chr,optional,phi.km)
write
}
for(i in 1:flag1){
  temp1[,flag] <- write(chr(flag,1)-chr(1,100)
}
if(temp1$phi,flag)==000{
  flag <- flag+1
  write
}
else{
  write(flag,1) <- write(0,phi,chr,optional,phi.km)
}
}
if(write==nil) warning("Data results is denied")
if(write==00000)
  stop("Program failed to accomplish your request of 40000 trials")
cat
}

```

program ends

 ##### sample run with the following parameters: #####
 ##### n= 1000, phi= 0, flag= 0, and write? #####
 #####

in option(10,10,0,0,0)

 the results of sample run are
 #####

	L,17	L,18	L,19	L,20	L,21	L,22	L,23	L,24	L,25
(1,1)	60	0.0	0.0	18	10	34	3	0.0	7000000
(2,1)	60	0.0	0.0	14	0	1	13	14.0	7007000
(3,1)	60	0.0	0.0	3	10	37	0	13.0	7007000
(4,1)	60	0.0	0.0	7	10	31	11	3.0	6013000
(5,1)	60	0.0	0.0	18	0	0	13	7.0	7007000
(6,1)	60	0.0	0.0	4	13	30	10	6.0	7007000
(7,1)	60	0.0	0.0	7	16	3	27	0.0	6013000
(8,1)	60	0.0	0.0	4	13	0	10	0.0	6000000
(9,1)	60	0.0	0.0	7	13	10	4	12.0	7007000
Col,1	60	0.0	0.0	94	0	10	3	10.0	7007700

column 1 is for θ

column 2 is for ϕ

column 3 is for ψ

column 4 is for α

column 5 is for β

column 6 is for γ

column 7 is for δ

column 8 is for ϵ

column 9 is for the actual p-values for a pG-generated design

#####

Remarks

- (1) In order to generate a design for an individual entry model, it is only needed to specify N and α values. A p value does not need to generate a design. The p value is used when to calculate the values of the quantities of interest, which are considered in this description.
- (2) When this program does not running properly as it takes too long to wait. I recommend to increase the size of "N" or to decrease the size of "want", or to increase the size of "D".

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BIOGRAPHICAL SKETCH

Junsung Lee was born on December 28, 1945, in Chicago City, Chongbuk Province, Republic of Korea, the first son of Choon-Ho Lee and Hye-Sook Cha. He received his bachelor's degree in statistics at Korea University (Seoul, Korea) in 1965. He also earned a master's degree in statistics at Korea University in 1968. He served as a second lieutenant in the Korean Army from 1965 to 1968. After an honorable discharge, he entered the University of California at Los Angeles in 1969 to study biostatistics, and received a master's degree in 1970. After he worked for a year in the Department of Family Medicine, UCLA, as a member of the research staff and also in the BAXCO corporation, Santa Monica, California, as a statistician, he entered the University of Florida to continue his study in statistics in 1974. Under the direction of Professor Asotul J. Kline, he received a Doctor of Philosophy degree in statistics in August 1979.

He has a sister, Sun-Hye Lee, and a brother-in-law, Min Cha, who have two beautiful daughters, Jo-Young and Seul-Young. He also has a brother, Sangdon Lee, and a sister-in-law, Seokyeon Kim. He has been married to Min-Jeong Kim since 1999. They have three wonderful, but mischievous, sons: Sang Hwan (David), Sun Hwan (John), and Yoon Hwan (Dante).

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.


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